RECURRENT SOLUTIONS OF DIFFERENTIAL EQUATIONS

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Abstract

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1°. Everywhere in what follows, by the letter T we shall denote the real line.

Let R be a metric space, r the distance in it, and φ a continuous mapping from T into R. Following the definition of recurrent motions introduced by G. D. Birkhoff ([1]), we shall call the mapping φ **recurrent** (**completely recurrent**) if for every $\varepsilon > 0$ (for every segment S and every $\varepsilon > 0$) there exists an $l > 0$ such that for every $\tau_0 \in T$, on any interval of length l there is a number τ for which

$$r(\varphi(\tau), \varphi(\tau_0)) < \varepsilon \quad \left(\sup_{t\in S} r(\varphi(t+\tau), \varphi(t+\tau_0)) < \varepsilon\right).$$

It is clear that every completely recurrent mapping is recurrent. The converse assertion is not true. However, if a recurrent mapping φ is a motion in some dynamical system, then it is completely recurrent. In this case φ is a recurrent motion in the sense of G. D. Birkhoff. Certain problems in the theory of recurrent functions were considered by V. I. Zubov ([2,3]).

G. D. Birkhoff showed that in every compact dynamical system there exists a recurrent motion. This result is one of the fundamental ones in the topological theory of dynamical systems ([4,5]). The question of the existence of recurrent solutions of differential equations is also of interest. The problem of finding such solutions was posed by V. V. Nemytskii (see ([6]), p. 101).

In the present note some criteria are established for the existence of completely recurrent solutions of the differential equation

$$dx/dt = f(x, t), \tag{1}$$

defined in a Banach space. These criteria show that, along with differential equations defining compact dynamical systems, a broad class of equations of the form (1), which, generally speaking, do not define dynamical systems, possess completely recurrent solutions.

V. M. Millionshchikov ([7]) showed that if the set of all solutions of equation (1), specified in a separable locally convex space and defined on infinite intervals of the form $(t_0, +\infty)$ and lying in some compact set (this set is assumed nonempty), is completed by all possible left shifts of them and then closed in the sense of uniform convergence of continuous mappings on segments, then in the completion constructed in this way there is a recurrent mapping. This mapping, generally speaking, is not a solution of the given equation and is not even a limit of such solutions. However, as Theorem 1 of the present note shows, if the right-hand side of equation (1) is uniformly, with respect to the variable x, completely recurrent, then in the completion of the solutions of this equation, constructed as indicated above,

method, there will be found a completely recurrent mapping that is its solution.

2°. Let us consider some characteristic properties of completely recurrent mappings. First we introduce the concept of a system of shifts, which will play an essential role in what follows.

Let R be a metric space, and let r be the distance in it. By a **system of shifts** in R we shall mean the dynamical system (Φ, Δ), where Φ is the space of all continuous mappings T into R, in which the distance ρ is defined by the formula

$$\rho(\varphi, \psi) = \sup_{\varepsilon > 0} \min \left\{ \sup_{|t| \leq 1/\varepsilon} r(\varphi(t), \psi(t)); \varepsilon \right\}$$

for any $\varphi, \psi \in \Phi$, and Δ is a mapping of the product $\Phi \times T$ into Φ, which for all $\varphi \in \Phi$ and $t \in T$ is defined by the condition $\Delta(\varphi, t) = \varphi^t$, where φ^t is the shift to the left of the mapping φ by t (we shall use this notation for left shifts of mappings also in what follows).

Let $\varphi, \psi \in \Phi$. The inequality $\rho(\varphi, \psi) < \varepsilon$ holds if and only if

$$\sup_{|t| \leq 1/\varepsilon} r(\varphi(t), \psi(t)) < \varepsilon.$$

It follows from this that convergence in the space Φ is uniform convergence on every interval. Moreover, Φ is complete if and only if R is complete.

The system of shifts (Φ, Δ) defined above is a generalization of the universal dynamical system of M. V. Bebutov ([8]), which is a system of shifts for T. The motion in the dynamical system (Φ, Δ) determined by an arbitrary point $\varphi \in \Phi$ will be denoted by the symbol Δ_φ.

Lemma 1. *A continuous mapping of the real line T into the space R is completely recurrent if and only if, in the system of shifts (Φ, Δ) over R, the motion Δ_φ is recurrent.*

Remark 1. In the topological theory of dynamical systems, along with the recurrent motions introduced by G. D. Birkhoff, stationary, periodic, and almost periodic motions are also of great interest. It can be shown that in the dynamical system (Φ, Δ) defined above the motion Δ_φ is stationary (periodic, almost periodic) if and only if the mapping φ is constant (periodic, almost periodic*).

Lemma 2. *Let the space R be complete, and let φ be a continuous mapping of T into R. The following conditions are equivalent:*

1. *The mapping φ is completely recurrent.*
2. *The set $\{\varphi(t);\ t \in T\}$ is completely bounded, and the mapping φ is uniformly continuous and has the following property: for every $\varepsilon > 0$ there exists a relatively dense set A such that for each $\tau \in A$ the inequality holds*

$$\sup_{|t| \le 1/\varepsilon} r(\varphi(t+\tau), \varphi(t)) < \varepsilon.$$

3. *For any sequence of numbers τ_m one can choose a subsequence τ_{m_n}, a completely recurrent mapping $\psi : T \to R$, and a sequence τ'_n such that the equalities*

$$\lim_{n\to\infty} \sup_{t\in s} r(\varphi(t+\tau_{m_n}), \varphi(t)) = 0,$$

$$\lim_{n\to\infty} \sup_{t\in s} r(\psi(t+\tau'_n), \varphi(t)) = 0$$

hold, whatever the interval s may be.

Lemma 2 can be proved without particular difficulty if one uses the concept of a system of shifts and some facts from the theory of dynamical systems.

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* The definition of an almost periodic function taking values in an arbitrary metric space is analogous to the definition given by H. Bohr ([9]) for real functions (see ([10])).

3°. Let us establish some criteria for the existence of completely recurrent solutions of the differential equation (1). Here it is assumed throughout that f is a continuous mapping of the product $D \times T$ into the Banach space B, and $D \subseteq B$.

We shall call the given mapping f **uniformly completely recurrent** if, for every segment s and every $\varepsilon > 0$, there exists an $l > 0$ such that, whatever $\tau_0 \in T$ may be, in every interval of length l there is a number τ for which

$$\sup_{x\in D,\ t\in s} \|f(x, t+\tau) - f(x, t+\tau_0)\| < \varepsilon.$$

It is easy to see that the mapping f is uniformly completely recurrent if, for example, it is uniform with respect to $x \in D$, almost periodic in $t \in T$, and, a fortiori, if it does not depend on the variable $t \in T$.

A solution φ of equation (1) will be called **completely recurrent** if φ is a completely recurrent mapping of T into B.

Theorem 1. *Let the set D be compact, and let the mapping f be uniformly completely recurrent. In order that there exist a completely recurrent solution of equation (1), it is necessary and sufficient that there exist a solution of this equation defined on an infinite interval.*

We give the outline of the proof of Theorem 1. Suppose that there exists a solution φ_0 of equation (1) defined on an infinite interval I. For definiteness we shall assume that $I = [t_0, +\infty)$.

Denote by (Φ, Δ) the system of shifts on D (we note that the space Φ is complete). Define the mapping $\varphi : T \to D$ by the relation

$$\varphi(t) = \begin{cases} \varphi_0(t), & \text{for } t > t_0, \\ \varphi_0(t_0), & \text{for } t \leq t_0, \end{cases}$$

and in the dynamical system (Φ, Δ) consider the motion Δ_φ. The mapping φ is uniformly continuous. Since, moreover, the set $\{\varphi(t);\, t \in T\}$ is completely bounded in D, the motion Δ_φ is stable in the sense of Lagrange (see (5), Lemma 1.65). Therefore the set Ω_φ of all ω-limit points of the motion Δ_φ contains some compact minimal set Ψ (see (4), p. 402, Corollary 2). Choose arbitrarily $\psi \in \Psi$. Since $\psi \in \Omega_\varphi$, there is a sequence of positive numbers τ_m such that

$$\lim_{m\to\infty} \tau_m = +\infty, \tag{2}$$

$$\lim_{m\to\infty} \Delta_\varphi(\tau_m) = \psi. \tag{3}$$

For the sequence τ_m thus obtained one can choose a subsequence τ_{m_n}, a uniformly continuous mapping g of the product $D \times T$ into B, and a sequence τ'_n such that

$$\lim_{n\to\infty} \sup_{x\in D,\ t\in s} \|f(x, t + \tau_{m_n}) - g(x,t)\| = 0, \tag{4}$$

$$\lim_{n\to\infty} \sup_{x\in D,\ t\in s} \|g(x, t + \tau'_n) - f(x,t)\| = 0, \tag{5}$$

whatever the segment s may be (cf. item 3 of Lemma 2). Moreover, from (3) it follows that

$$\lim_{n\to\infty} \sup_{t\in s} \|\varphi(t+\tau_{m_n}) - \psi(t)\| = 0, \tag{6}$$

whatever the segment s may be. Taking into account equalities (2), (4), (6), and the fact that for every natural n the restriction of the mapping $\varphi^{\tau_{m_n}}$ to the set $[t_0 - \tau_{m_n}, +\infty)$ is a solution of the equation $dx/dt = f(x, t + \tau_{m_n})$, defined on the interval $[t_0 - \tau_{m_n}, +\infty)$, we conclude that ψ is a solution of the differential equation $dx/dt = g(x, t)$, defined on T.

The mapping ψ is completely recurrent. Using item 3 of Lemma 2, for the sequence τ'_n choose a subsequence τ'_{n_i} and a completely recurrent mapping $\chi : T \to D$ such that the equality

$$\lim_{i\to\infty} \sup_{t\in s} \|\psi(t+\tau'_{n_i}) - \chi(t)\| = 0 \tag{7}$$

holds, whatever the segment s may be. For any natural i, the mapping $\psi^{\tau'_{n_i}}$ is a solution of the differential equation

$$dx/dt = g(x, t + \tau'_{n_i}),$$

defined in T. Taking (5) and (7) into account, we conclude that χ is a solution of (1).

Remark 2. For the differential equation (1) a number of criteria are known for the continuability of solutions to infinite intervals (see, for example, ([11]), Theorems 3 and 4). According to Theorem 1, in the case when D is compact and the mapping f is uniformly completely recurrent, each of them is a criterion for the existence of a completely recurrent solution of the indicated equation.

Following the terminology adopted in the theory of dynamical systems, we shall say that a solution φ of the differential equation (1), defined on an interval I, is **Lagrange stable in the positive (negative) direction** if one can choose $t_0 \in T$ and a compact set $C \subseteq D$ such that $[t_0, +\infty) \subseteq I$ ($(-\infty, t_0)$ and $\varphi(t) \in C$ for all $t \geq t_0$ ($t \leq t_0$). We shall call the given solution φ **Lagrange stable** if it is Lagrange stable both in the positive and in the negative direction.

The following proposition is established on the basis of Theorem 1.

Theorem 2. *Suppose that for every compact $C \subseteq D$ the restriction of the mapping f to $C \times T$ is uniformly completely recurrent. Then, if there exists a solution φ of equation (1), defined on some interval I, which is Lagrange stable in at least one direction, then there exists a solution χ of the same equation, defined in T, which is completely recurrent and for which $\chi(T) \subseteq \varphi(I)$.*

Let us formulate some consequences of Theorem 2.

Corollary 1. *Suppose that for every compact $C \subseteq D$ the restriction of the mapping f to $C \times T$ is uniformly completely recurrent. Then, if φ is the unique Lagrange-stable solution of equation (1), then it is completely recurrent. Moreover, if D is closed in B, then φ is the unique completely recurrent solution of the given equation.*

Corollary 2. *Let A be a real matrix of order n, whose spectrum does not intersect the imaginary axis. Whatever completely recurrent mapping $\alpha : T \to T^n$ is given, there exists a unique completely recurrent solution of the equation $dx/dt = Ax + \alpha(t)$.*

The last proposition is established with the aid of Theorem 2.3 of work ([12]).

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[1] G. D. Birkhoff, *Dynamical Systems*, Moscow, 1941. [2] V. I. Zubov, DAN, **132**, No. 3 (1960). [3] V. I. Zubov, *Siberian Mathematical Journal*, **3**, No. 4 (1962). [4] V. V. Nemytskii, V. V. Stepanov, *Qualitative Theory of Differential Equations*, Moscow–Leningrad, 1949. [5] W. H. Gottschalk, G. A. Hedlund, *Topological Dynamics*, Providence, **36**, 1955. [6] V. V. Nemytskii, UMN, **4**, issue 6 (34) (1949). [7] V. M. Millionshchikov, DAN, **161**, No. 1 (1965). [8] M. Bebutov, DAN, **27**, No. 9 (1940). [9] H. Bohr, *Almost Periodic Functions*, Moscow–Leningrad, 1934. [10] Tornehave, *12-te Skandinaviska matematiker kongressen*, 1953. [11] M. A. Krasnosel' skii, S. G. Krein, *Transactions of Voronezh University*, issue 2 (1956). [12] M. G. Krein, *Lectures on the Theory of Stability of Solutions of Differential Equations in Banach Space*, Kiev, 1964.

Note: Figure translations are in progress. See original paper for figures.