HOMEOMORPHIC MAPPINGS IN THREE-DIMENSIONAL EUCLIDEAN SPACE THAT PRESERVE ANGLES FOR RAYS

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Abstract

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A. P. KOPYLOV

HOMEOMORPHIC MAPPINGS IN THREE-DIMENSIONAL EUCLIDEAN SPACE THAT PRESERVE ANGLES FOR RAYS

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In paper (1), D. E. Men' shov proved a theorem important for the geometric characterization of a planar conformal mapping (D. E. Men' shov gave it its most general form in (2)).

Theorem 1. *If a homeomorphic mapping $w = f(z)$ of a plane domain G onto a plane domain G' preserves, at every point z of G, the angles for all pairs of curves with tangents at the point z, then $f(z)$ is a holomorphic transformation in G.*

The same theorem admits the following equivalent formulation, in which the condition is reduced to the requirement of uniform preservation of angles at a point:

Theorem 1′. *If $w = f(z)$ is a homeomorphic transformation of a plane domain G, and if for every positive number ε and every point $z \in G$ one can indicate a number $\delta = \delta(\varepsilon, z) > 0$ such that*

$$|\angle f(z')f(z)f(z'') - \angle z'zz''| < \varepsilon$$

for all points z' and z'' from the δ-neighborhood of the point z, then $f(z)$ is a holomorphic transformation in G.

For the case of three-dimensional Euclidean space, Theorem 1 (1′) is also valid; moreover, its proof can be carried out by the methods of D. E. Men' shov, and therefore from this point of view it is less interesting.

Our aim is to dispense with uniformity (we require preservation of angles only for rays). In achieving this aim we had to overcome difficulties connected with the specific features of three-dimensional space, whose essence reduces to the fact that a curve in space does not bound a volume.

The proof of Theorem 3 is also valid in the planar case and constitutes a new proof of assertions of this kind for planar mappings.

Definition. Let $y = f(x)$ be a homeomorphic mapping of a domain D of three-dimensional space onto a domain D' of the same space. We shall say that $f(x)$ **preserves angles for rays at the point** $x_0 \in D$ if every pair of linear segments x_0x_1 and x_0x_2 located in D is transformed under the mapping $f(x)$ into a pair of curves possessing tangent rays T_1 and T_2 at the point $y_0 = f(x_0)$, and the least positive angle between the rays $\overrightarrow{x_0x_1}$ and $\overrightarrow{x_0x_2}$ is equal to the analogous angle between T_1 and T_2. If $f(x)$ preserves angles for rays at every point $x \in D$, then we shall call it **angle-preserving for rays in** D.

Theorem 2. *An angle-preserving-for-rays homeomorphic mapping $y = f(x)$ of a three-dimensional domain D into three-dimensional space is a Möbius transformation of this domain.*

The **proof** consists of four parts.

1. Differentiability of $f(x)$ almost everywhere in D.
2. Existence everywhere of a dense in D and open set of points of Möbiusness (we call a point $x \in D$ a **point of Möbiusness of the map–**

image $f(x)$, if there exists a ball V with center at x and belonging to D, in which $f(x)$ is a Möbius transformation).

3. **Proof of the assertion.** Let λ be an arbitrary plane in three-dimensional space, P, $\bar{P} \subset D$ a set closed in D, at each point of which $y = f(x)$ preserves angles for rays, while in $D - P$, $f(x)$ is a locally Möbius transformation. Then the subset of those points of P for which the image of the plane λ_x, passing through the point x parallel to λ, has a tangent plane at the point $y = f(x)$, is everywhere of second category on P.

4. **The erasure of a certain portion of the set P of points which are not points of Möbiusness.**

Almost everywhere in D, the differentiability of the transformation $y = f(x)$ follows from the following assertion:

Theorem 3. *Let $y = f(x)$ be a homeomorphic mapping of a domain D in three-dimensional space, and suppose that at every point of a measurable set $E \subset D$, $mE > 0$, $f(x)$ preserves angles for rays. Then almost everywhere on E, $f(x)$ transforms the family of infinitely small spheres into a family of surfaces with bounded distortion relative to the image of the center of the spheres, i.e.*

$$\lim_{r \to 0} \frac{\left[\max_{|x'-x|=r} |f(x') - f(x)|\right]}{\left[\min_{|x'-x|=r} |f(x') - f(x)|\right]} < \infty$$

almost everywhere on E.

The idea of the proof of this theorem is as follows. The contrary assumption, taking into account the measurability of the set of those points of D at which the family of infinitely small spheres is transformed into a family of surfaces with bounded distortion relative to the image of the center of the spheres, leads to the existence of a closed bounded set F, $mF > 0$, $F \subset D$, at every point of which $f(x)$ preserves angles for rays and for every point x of which one can indicate two sequences of points $\{x'_n\} = \{x'_n(x)\}$ and $\{x''_n\} = \{x''_n(x)\}$ such that $|x'_n - x| = |x''_n - x|$, $n = 1, 2, \ldots$, $x'_n \to x$ and

$$|f(x'_n) - f(x)|/|f(x''_n) - f(x)| \xrightarrow[n\to\infty]{} \infty.$$

Using the property of preservation of angles for rays, the continuity of the transformation $y = f(x)$, Baire' s category theorem, and also the existence of sequences of points $\{x'_n(x)\}$ and $\{x''_n(x)\}$ for points x of F, we find a point $x_0 \in F$ such that:

1) there exists a sequence $\{R_n\}$ of somewhat deformed isosceles right triangles contracting to the point x_0, the vertex of the right angle of each of which is the point x_0, and which under the mapping $y = f(x)$ pass into curvilinear triangles R'_n, $n = 1, 2, \ldots$, differing only slightly from isosceles right triangles, with the legs and the hypotenuse of the triangle R_n ($n = 1, 2, \ldots$) passing respectively into the legs and the hypotenuse of the triangle R'_n;

2) the hypotenuse of the triangle R_n, $n = 1, 2, \ldots$, contains a pair of points $\tilde{x}'_n$ and $\tilde{x}''_n$ such that

$$|f(\tilde{x}'_n) - f(x_0)|/|f(\tilde{x}''_n) - f(x_0)| \xrightarrow[n\to\infty]{} \infty.$$

But since for any two points y'_n and y''_n of the hypotenuse of the triangle R'_n

$$|y'_n - y_0|/|y''_n - y_0| < C \qquad (y_0 = f(x_0)),$$

where C depends neither on the chosen pair of points nor on n, 2) contradicts 1).

This same idea is characteristic of the proof of Theorem 2 at all its stages, except for the last part. The fourth part is proved by the methods of D. E. Menshov.

Lviv State University
named after Ivan Franko

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CITED LITERATURE

[1] D. Menchoff, *Math. Ann.*, **95**, 640 (1926). [2] D. Menchoff, *Math. Ann.*, **109**, 101 (1933).