ANALOGY OF SYSTEMS OF DIFFERENTIAL EQUATIONS NEAR A SINGULAR POINT

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Abstract

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ANALOGY OF SYSTEMS OF DIFFERENTIAL EQUATIONS NEAR A SINGULAR POINT

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1°. Here we shall consider the systems

$$x' = Ax + F(t, x), \tag{1}$$

$$y' = Ay, \tag{2}$$

where A is a matrix with constant coefficients of order n; x, y, F are n-dimensional vectors; $F(t, x)$ is given on the cylinder $t \geq 0,\ |x| \leq r_0 < 1$*. It is assumed that

$$F(t, \theta) = 0, \tag{3}$$

$$|F(t, x_1) - F(t, x_2)| \leq L(r)|x_1 - x_2|, \tag{4}$$

where $r = \max(|x_1|, |x_2|)$; $L(r)$ is a monotonically increasing function on $[0; r_0]$.

2°. Denote by $\omega_1, \omega_2, \dots, \omega_s$ the distinct real parts of the eigenvalues of the matrix A, and by S a neighborhood of the point $x = 0$ of radius r_0. We shall assume that $\omega_1 < \omega_2 < \cdots < \omega_s$ and that $\omega_1 < 0$. Obviously, the solutions of system (2) with negative exponents** fill some subspace $\overline{R}$ of the n-dimensional space. Let the set $\overline{S}$ consist of those points of the neighborhood S through which, at the moment $t = 0$, pass solutions of system (1) with negative exponents.

Theorem 1. *If $L(r) \to 0$ as $r \to 0$, $\omega_1 < 0$, then the set $\overline{S}$ contains points distinct from the origin. Moreover, there exists a certain neighborhood S^* of the point $x = 0$ and its homeomorphic mapping Φ onto a certain region, possessing the following properties:*

1. The image of the set $\overline{S}^* = S^* \cap \overline{S}$ under Φ lies in the subspace $\overline{R}$ and contains all points of $\overline{R}$ with sufficiently small norms; $\Phi(0) = 0$.

2. Through the points of $\overline{S}^*$ *and* $\overline{R}$ *corresponding under* Φ*, at the moment* $t = 0$*, pass solutions* $x(t)$ *and* $y(t)$ *of systems (1) and (2) with identical exponents; moreover, if the exponent of* $x(t)$ *and* $y(t)$ *is equal to* $\omega_k < 0$*, then for* $t \geq 0$

$$|x(t) - y(t)| \leq |x(0)| f(|x(0)|, \varepsilon) e^{(\omega_k + \varepsilon)t},$$

where $\varepsilon > 0$ *is arbitrarily small,* $f(r, \varepsilon) \to 0$ *as* $r \to 0$*.*

3. The mappings Φ *and* Φ^{-1} *satisfy the Lipschitz condition and have the form*

$$\Phi(x) = x + \varphi(x), \qquad \Phi^{-1}(x) = x + \psi(x),$$

where

$$|\varphi(x)| = o(|x|) \qquad \text{as } x \to 0,$$

$$|\psi(x)| = o(|x|) \qquad \text{as } x \to 0,$$

$$|\varphi(x)| / |\psi(x)| \to 0, \qquad \text{when } x \to 0.$$

* $|x|$ is the Euclidean norm of the vector x.

** The exponent of a vector $z(t)$ $(t \geq t_0)$ is defined to be $\overline{\lim}_{t\to\infty} \dfrac{1}{t} \ln |z(t)|$.

If among the numbers $\omega_1, \omega_2, \ldots, \omega_s$ there is none equal to zero and $L(r)$ is sufficiently small, then from the points of the set $S \setminus \overline{S}$ at the moment $t = 0$ there issue solutions of system (1) that leave the neighborhood S for finite values of t.

Moreover, every solution $x(t)$ of system (1) beginning at a point $x_0 \in S^*$ satisfies the inequality

$$|x(t)| \leq C_\varepsilon |x_0| e^{(\omega + \varepsilon)t},$$

where $\varepsilon > 0$ is arbitrarily small, C_ε depends on ε and does not depend on x_0, and ω is the exponent of $x(t)$.

3°. Let the function $L(r)$ be given by the equalities:

$$L(0) = 0, \tag{5}$$

$$L(r) = \varepsilon(r) r^\lambda |\ln r|^\mu \quad \text{for } r \in (0, r_0],$$

where $\varepsilon(r)$ is a monotonically increasing function on $[0, r_0]$ satisfying the condition

$$\int_0^{r_0} \frac{\varepsilon(r)}{r|\ln r|}\, dr < +\infty, \tag{6}$$

$\lambda \geq 0$ and μ are certain numbers.

We note that as $\varepsilon(r)$ one may take the functions $|\ln r|^{-\delta}$ or r^{δ}, where $\delta > 0$.

Consider all boxes of the Jordan form A that correspond to eigenvalues with real parts equal to $\omega_k < 0$. Denote by $m_k + 1$ the largest of the orders of these boxes and introduce the number $\omega_0 < \omega_1(1+\lambda)$. For each index k such that $\omega_k < 0$, find the number $\tilde{k}$ from the inequalities

$$\omega_k(1+\lambda) > \omega_{\tilde{k}-1}, \tag{7}$$

$$\omega_k(1+\lambda) \leq \omega_{\tilde{k}}.$$

Put

$$m^c_{\tilde{k}} = \begin{cases} 0, & \text{if } \omega_k(1+\lambda) < \omega_{\tilde{k}}, \\ m_{\tilde{k}}, & \text{if } \omega_k(1+\lambda) = \omega_{\tilde{k}}. \end{cases} \tag{8}$$

For an arbitrary vector $x(t)$ with (finite) exponent ω, define the number

$$l = \overline{\lim}_{t\to\infty} \frac{\ln\left[e^{-\omega t}|x(t)|\right]}{\ln t},$$

which we shall call the **second exponent** of the vector $x(t)$. It is clear that the second exponents of solutions of linear systems with constant coefficients are nonnegative integers. For convenience we shall sometimes call the exponent the first exponent.

The fraction $|x(t) - y(t)|/|y(t)|$ may be regarded as the relative error when the "exact" vector $x(t)$ is replaced by the "approximate" vector $y(t)$. We shall call this fraction the **deviation** of x from y. If the deviation of x from y tends to 0 as $t \to \infty$, then we shall say that x and y are **analogous**.

It is easy to see that analogous vectors have not only the same exponents, but also the same second exponents. Moreover, the ratio of the norms of analogous vectors tends to 1 as $t \to \infty$, while the differences between the directing cones tend to zero.

Introduce the function $\rho(t)$:

$$\rho(t) = \begin{cases} 1, & \text{if } |t| < 1, \\ |t|, & \text{if } |t| \geq 1. \end{cases}$$

4°. **Theorem 2.** Let the function $L(r)$ be defined by formula (5), condition (6) be satisfied, $\lambda = 0$, $\mu < 0$.

Then:

1. All assertions of Theorem 1 are valid.
2. Every solution $x(t)$ of system (1) with exponent $\omega_k < 0$, where the index k is such that $\mu \leq -(1+m_k)$, is analogous to some solution $y(t)$ of system (2), and the deviation of x from y is $o(t^{m_k+1+\mu})$ as $t \to \infty$.

Moreover, if the point $x(0) \in S^*$ (see Theorem 1), then for the solution $x(t)$ the estimate

$$|x(t)| \leq M|x(0)|e^{\omega_k t}\rho^l(t), \qquad t \geq 0, \tag{9}$$

holds, where $M > 0$ does not depend on $x(0)$, and l is the second exponent of $x(t)$.

3. On S^* there is defined a homeomorphism Φ^*, possessing all the properties of the homeomorphism Φ indicated in Theorem 1, and such that if $\mu \leq -(1+m_k)$, then the solutions $x(t)$ and $y^*(t)$ of systems (1) and (2), passing at the moment $t = 0$ through the corresponding (by virtue of Φ) points and having exponents equal to $\omega_k < 0$, are analogous, and their deviation is $o(t^{m_k+1+\mu})$ as $t \to \infty$; and for them, for $t \geq 0$, the inequality

   $$|x(t) - y^*(t)| \leq |x(0)|\psi(t, |x(0)|)e^{\omega_k t}\rho^{l+m_k+1+\mu}(t),$$

   holds, where l is the second exponent of x and y; $\psi(t, r) \to 0$ when $t + r^{-1} \to \infty$.

Theorem 3. Suppose that the conditions of Theorem 2 are satisfied, with the sole difference that $\lambda > 0$, and μ is arbitrary.

Then:

1. All assertions of Theorem 1 are valid.
2. Every solution $x(t)$ of system (1) with exponent $\omega_k < 0$ is analogous to some solution $y(t)$ of system (2), and their deviation is

   $$o\left(e^{\lambda\omega_k t}t^{m_k^0+1+\lambda l+\mu}\right)$$

   as $t \to \infty$, where l is their second exponent.
3. For every solution $x(t)$ of system (1) with exponent $\omega_k < 0$ such that $x(0) \in S^*$, inequality (9) holds.

4. In a neighborhood of S^* there is defined a homeomorphism Φ^*, having the properties of the homeomorphism Φ listed in Theorem 1, and such that the solutions $x(t)$ and $y^*(t)$, passing at $t = 0$ through the corresponding, by virtue of Φ, points of the sets S^* and $\bar{R}$, prove to be analogous, and their deviation is
$$o\Big(e^{\lambda\omega_k t} t^{m_k^0+1+\lambda l+\mu}\Big)$$
as $t \to \infty$, where ω_k and l are their first and second exponents.

Moreover, for $t \geq 0$,

$$|x(t) - y^*(t)| \leq |x(0)|^{1+\lambda}\psi(t, |x(0)|)e^{(1+\lambda)\omega_k t}\rho^{(1+\lambda)l+m_k^0+1+\mu}(t),$$

where, in the case $\mu \leq 0$, $\psi(t, r) \to 0$ when $t + r^{-1} \to \infty$, while if $\mu > 0$, then $\psi(t, r) = |\ln r|^{\mu}\varepsilon(t, r)$, where $\varepsilon(t, r) \to 0$ when $t + r^{-1} \to \infty$.

5. In addition to property 3 of the homeomorphism Φ (Theorem 1), one may assert that

$$\Phi^*(x) = x + \varphi^*(x), \qquad \Phi^{*-1}(x) = x + \psi^*(x),$$

where
$$|\varphi^*(x)| = o(|x|^{1+\lambda}) \quad \text{as } x \to 0, \quad \text{if } \mu \leq 0;$$
$$|\varphi^*(x)| = o(|\ln|x||^{\mu}|x|^{1+\lambda}) \quad \text{as } x \to 0, \quad \text{if } \mu > 0;$$
$$|\varphi^*(x)|/|\psi^*(x)| \to 1, \qquad \text{when } x \to 0.$$

5°. From Theorems 1 and 2 one can, with the aid of the principle of linear inclusion (1), derive conclusions about the behavior of solutions of system 1 also in the case when the vector F, instead of conditions (3) and (4), satisfies the condition

$$|F(t, x)| \leq L(|x|)|x|. \tag{10}$$

Without formulating the results precisely, let us say only that if $L(r)$ satisfies the condition of one of Theorems 1-3, then for solutions of systems (1) and (2) all estimates of this theorem are valid, although there may be no homeomorphism.

It should be noted, however, that when condition (10) is fulfilled, it is possible to prove the existence of solutions of the same type as under conditions (3) and (4). For example, the following proposition is true:

If $\lambda = 0$, $\mu \leq -(1 + m_k)$, $\omega_k < 0$, *and inequality (10) is fulfilled, then every solution of system (1) with exponent* ω_k *is analogous to some solution of system (2), and conversely, every solution of system (2) with exponent* ω_k *is analogous to some solution of system (1).*

We note that in paper (2) there are theorems close to the proposition stated above, and that Theorems 2 and 3 refine the results presented in papers (3–5).

One can give examples showing that abandonment of requirement (6) entails either the appearance of nonanalogous solutions (under the conditions of Theorem 2), or an increase in the order of deviation (in comparison with that guaranteed by Theorem 3). At the same time there exist systems of the form (1) for which conditions (3), (4), (5), and (6) are satisfied, and the deviations of whose solutions from the solutions of the corresponding linear system are quantities precisely of the order guaranteed by Theorems 2 and 3.

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CITED LITERATURE

[1] B. F. Bylov, D. M. Grobman, UMN, vol. 3 (105), 159 (1962).
[2] P. Hartman, A. Wintner, Am. J. Math., 77, No. 4, 692 (1955).
[3] I. G. Petrovskii, Matem. sborn., 41, 107 (1934).
[4] E. A. Kodlington, N. Levinson, *Theory of Ordinary Differential Equations*, IL, 1958, pp. 343–375.
[5] D. M. Grobman, DAN, 108, No. 4, 571 (1956).

Note: Figure translations are in progress. See original paper for figures.