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ON PARABOLIC EQUATIONS WITH CONTINUOUS COEFFICIENTS

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Abstract

Full Text

MATHEMATICS

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ON PARABOLIC EQUATIONS WITH CONTINUOUS COEFFICIENTS

(Presented by Academician I. G. Petrovsky, 28 II 1966)

In the present note a negative answer is given to the following closely related questions: 1) Does there exist a classical solution of the simplest boundary-value problems for any linear parabolic equation

$$\frac{\partial u}{\partial t} = \sum_{i,j=1}^n a_{ij}(x,t) \frac{\partial^2 u}{\partial x_i \partial x_j} \quad (1)$$

with continuous coefficients? 2) Does there exist a bounded solution of the equation adjoint to it? 3) Does a parabolic equation with self-adjoint principal part and continuous coefficients have a solution with bounded derivatives? The example constructed below also shows that it is impossible to obtain, in the metric C , a priori estimates of the second derivatives $\partial^2 u / \partial x_i \partial x_j$ for solutions of equation (1), depending only on the modulus of continuity of the coefficients and on the minimal eigenvalue of the matrix $\|a_{ij}\|$. Analogous conclusions are also valid for the two other types of equations mentioned above.

In the domain $G\{|x| \leq 1; 1/2 \leq t < 1\}$ consider the equation

$$\partial u / \partial t = a(x,t) \partial^2 u / \partial x^2, \quad (2)$$

where the function

$$a(x,t) = 2 + \{\ln(1-t-x^2)\}^{-1} \psi\{(1-t)^{-1/2}[1-|x|(1-t)^{-1/2}]\},$$

and the function $\psi(\xi) \in C_\infty(-\infty, \infty)$. Moreover $\psi'(\xi) \geq 0$, $\psi(\xi) \equiv 0$ for $\xi \leq \sigma_0$, and $\psi(\xi) \equiv m \geq 0$ for $\xi \geq \sigma > 0$. The absolute constant m is chosen so that everywhere in \bar{G} the coefficient $a(x,t) \geq 1$; the positive constants σ_0 and σ are chosen sufficiently small. The function $a(x,t)$ thus constructed is infinitely differentiable for $t < 1$ and continuous in \bar{G} ; however, its modulus of continuity with respect to x in the domain G is of order $(\ln h)^{-1}$. It is easy to show that the constant σ can be chosen so that

$$\partial a / \partial t \geq \psi / (1-t-x^2) [\ln(1-t-x^2)]^2.$$

If on the lower and lateral sides of the rectangle \bar{G} one prescribes any smooth values of the function $u(x, t)$ such that the compatibility conditions are satisfied, then in the domain G there exists a unique smooth solution of equation (2) (see, for example, (1)). Below it will be shown that the boundary conditions can be chosen in such a way that $\partial u/\partial t(0, t) \rightarrow \infty$ as $t \rightarrow 1$. It follows from this that, for these boundary conditions, in the rectangle \bar{G} (including the upper side of the rectangle) there does not exist a classical solution of equation (2), i.e., a solution possessing everywhere continuous or at least finite derivatives entering the equation.

We choose the boundary conditions so that the inequalities are satisfied:

$$\frac{\partial^2 u}{\partial x^2}(x, 1/2) \geq 1 \quad \text{and} \quad \frac{\partial u}{\partial t}(\pm 1, t) \geq 1.$$

By virtue of equation (2), the derivative $\partial u/\partial t \geq 1$ on the three sides of the rectangle G : $\{t = 1/2\}$ and $\{x = \pm 1\}$, the sum of which we denote by Γ . Denote the derivative $\partial u/\partial t$ by v . Differentiating equation (2) with respect to t , we obtain the equa-

where $\partial v/\partial t = a(x, t)(\partial^2 v/\partial x^2) + [a(x, t)]^{-1}(\partial a(x, t)/\partial t)v$. Since $v|_{\Gamma} \geq 0$, it follows from the maximum principle that $v(x, t) \geq 0$ everywhere in G . In view of the fact that $\partial a/\partial t \geq 0$, and $v|_{\Gamma} \geq 1$, one may conclude that $v(x, t) \geq 1$ everywhere in G .

In order to show that $v(0, t) \rightarrow \infty$ as $t \rightarrow 1$, we construct an auxiliary function $w_1(x, t)$ for $|x| \leq 1$, $t < 1$, in a neighborhood of the segment $t = 1$, possessing the following properties: $w_1(x, t)$ is continuous, $w_1(\pm 1, t)$ is bounded; the function $w_1(x, t)$ is smooth and satisfies the inequality

$$\begin{aligned} L_1 W_1 &= \frac{\psi\{(1-t)^{-1/2}[1-|x|(1-t)^{-1/2}]\}}{(1-t-x^2)\ln(1-t-x^2)^2} \equiv \\ &\equiv a \frac{\partial^2 w_1}{\partial x^2} - \frac{\partial w_1}{\partial t} + \frac{\psi\{(1-t)^{-1/2}[1-|x|(1-t)^{-1/2}]\}}{(1-t-x^2)[\ln(1-t-x^2)]^2} \geq 0 \end{aligned} \quad (3)$$

everywhere except for a finite number of smooth curves $x = x_i(t)$; on these curves

$$\frac{\partial w_1(x-0, t)}{\partial x} < \frac{\partial w_1(x+0, t)}{\partial x};$$

finally, $w_1(0, t) \rightarrow \infty$ as $t \rightarrow 1$. If it is possible to construct such a function $w_1(x, t)$ in some rectangle $G_1\{|x| \leq 1; t_1 \leq t < 1\}$, then for some small positive ε the function $v - \varepsilon w_1$ is positive on the lower and lateral sides of G_1 , and

$$L_1(v - \varepsilon w_1) \leq -\frac{1}{a} \frac{\partial a}{\partial t} + \varepsilon \frac{\partial a}{\partial t} \leq 0$$

everywhere except for the curves $x = x_i(t)$. By virtue of the maximum principle (see, for example, (1), p. 138), $v - \varepsilon w_1 \geq 0$ in G_1 . Consequently, $v(0, t) \rightarrow \infty$ as $t \rightarrow 1$, together with $w_1(0, t)$.

Thus it remains to construct the function $w_1(0, t)$. This is conveniently done in the new coordinates

$$\xi = x(1-t)^{-1/2}, \quad \tau = \ln(1-t),$$

putting $w_1(x, t) \equiv w(\xi, \tau)$. The domain G_1 passes into the domain

$$G_2\{|\xi| \leq e^{\tau/2}; \tau_1 \leq \tau < \infty\},$$

where τ_1 is sufficiently large. The function $w(\xi, \tau)$ must be bounded for $|\xi| = e^{\tau/2}$. Condition (3) passes into the relation

$$Lw + \frac{\psi[e^{\tau/2}(1-|\xi|)]}{(1-\xi^2)[\ln(1-\xi^2) - \tau]^2} \geq 0, \quad (4)$$

where $Lw \equiv aw_{\xi\xi} - w_{\tau} - \frac{1}{2}\xi w_{\xi}$. Inequality (4) must be fulfilled everywhere in G_2 , except for several smooth curves $\xi = \xi_i(\tau)$. On these curves

$$\frac{\partial w}{\partial \xi}(\xi - 0, \tau) < \frac{\partial w}{\partial \xi}(\xi + 0, \tau), \quad (5)$$

and, finally,

$$w(0, \tau) \rightarrow \infty. \quad (6)$$

We shall construct the function $w(\xi, \tau)$ even with respect to ξ . For $\xi \geq 1$ put

$$w(\xi, \tau) = \alpha \int_1^{\tau} \frac{1}{\theta} \frac{\exp\{-\frac{1}{8}\xi^2(e^{\tau-\theta} - 1)^{-1}\}}{\sqrt{1 - \exp(\theta - \tau)}} d\theta. \quad (7)$$

Here and in what follows we assume τ sufficiently large. The positive constant $\alpha \leq 1$ will be chosen later. The function (7) satisfies the equation

$$2w_{\xi\xi} - w_{\tau} - \frac{1}{2}\xi w_{\xi} = 0,$$

and therefore also relation (4), since $a = 2$ for $\xi \geq 1$. It is easy to verify that the function constructed is bounded for $\xi = e^{\tau/2}$ for all $\tau \geq 1$. Denote

$$\mu(\tau) = \int_1^{\tau} \frac{1}{\theta} \frac{\exp[-\frac{1}{8}(e^{\tau-\theta} - 1)^{-1}]}{\sqrt{1 - \exp(\theta - \tau)}} d\theta.$$

This is a smooth positive function. It is not hard to see that

$$\mu(\tau) > M_1 \ln \tau, \quad (8)$$

$$|\mu'(\tau)| < M_2/\tau. \quad (9)$$

We shall denote positive absolute constants by M_i . By construction, $w(1+0, \tau) = \alpha\mu(\tau)$; $\frac{\partial w}{\partial \xi}(1+0, \tau) > -\alpha M_3/\tau$.

For $1 - \sigma e^{-\tau/2} \leq \zeta \leq 1$, put

$$w(\zeta, \tau) = \alpha \left\{ \mu(\tau) + \frac{M_4}{\tau} [1 - \zeta + M_5(1 - \zeta)^2] \right\}.$$

The constants M_4 and M_5 are easily chosen so that relation (5) holds for $\zeta = 1$ and $Lw > 0$. We now denote

$$\frac{1}{\alpha} w(1 - \sigma e^{-\tau/2}, \tau) \equiv \mu(\tau) + \frac{M_4}{\tau} (\sigma e^{-\tau/2} + M_5 \sigma^2 e^{-\tau})$$

by $\mu_1(\tau)$. From relations (8) and (9) it follows that $\mu_1(\tau) \rightarrow \infty$ and $|\mu_1'(\tau)| < M_6/\tau$.

In the region $1 - e^{-\tau/4} \leq \zeta \leq 1 - \sigma e^{-\tau/2}$, put

$$w(\zeta, \tau) = \alpha\mu_1(\tau) + \frac{m}{8} \left[e^\tau \int_{\sigma \exp(-3/2\tau)}^{(1-\zeta)\exp(-\tau)} (\ln \xi)^{-1} d\xi + \frac{11}{15} (1 - \zeta - \sigma e^{-\tau/2}) \right].$$

In this region $\psi \equiv m$, so that

$$\begin{aligned} & Lw + \frac{\psi}{(1 - \zeta^2)[\ln(1 - \zeta^2) - \tau]^2} = \\ & = -\alpha\mu_1'(\tau) - \frac{m}{8} \frac{a}{(1 - \zeta)[\ln(1 - \zeta) - \tau]^2} + \frac{a}{(1 - \zeta^2)[\ln(1 - \zeta^2) - \tau]^2} + O\left(\frac{1}{\tau}\right) > 0. \end{aligned}$$

Denote

$$w_1(1 - e^{-\tau/4}, \tau) \equiv \alpha\mu_1(\tau) + \frac{m}{8} \left[e^\tau \int_{\sigma \exp(-3/2\tau)}^{\exp(-5/4\tau)} (\ln \xi)^{-1} d\xi + \frac{11}{15} (e^{-\tau/4} - \sigma e^{-\tau/2}) \right] \equiv \mu_2(\tau).$$

Obviously, $\mu_2(\tau) \rightarrow \infty$ as $\tau \rightarrow \infty$, and $|\mu_2'(\tau)| < \alpha M_6/\tau + o(1/\tau)$.

Finally, define the function $w(\zeta, \tau)$ for $|\zeta| \leq 1 - e^{-\tau/4}$. For these values of ζ , put

$$w(\zeta, \tau) = \mu_2(\tau) + \frac{\alpha}{\tau} \left\{ \operatorname{ch} M_7 \zeta - \operatorname{ch} [M_7(1 - e^{-\tau/4})] \right\}.$$

The constant M_7 can be chosen so that the inequality $Lw > 0$ holds for $\tau > \tau_2(\alpha)$. The function $w(\zeta, \tau)$, by construction, is continuous and satisfies relations (4) and (6). It remains only to satisfy inequalities (5). This can be achieved by choosing α sufficiently small, since, on the one hand,

$$\frac{\partial w}{\partial \zeta}(1 - \sigma e^{-\tau/2} + 0, \tau) > -\alpha M_4/\tau \quad \text{and} \quad \frac{\partial w}{\partial \zeta}(1 - e^{-\tau/4} - 0, \tau) < \alpha M/8\tau,$$

and, on the other hand,

$$\frac{\partial w}{\partial \zeta}(1 - \sigma e^{-\tau/2} - 0, \tau) = -\frac{m}{120\tau} + o(1/\tau), \quad \frac{\partial w}{\partial \zeta}(1 - e^{-\tau/4} + 0, \tau) = \frac{m}{120\tau} + o(1/\tau).$$

The construction of the example is complete.

Let us now consider the Einstein-Kolmogorov differential equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2}{\partial x^2}(a(x, t)u) \tag{10}$$

with continuous coefficient $a(x, t)$. After the substitution $a(x, t)u = v$, it becomes the equation

$$\frac{1}{a} \frac{\partial v}{\partial t} + \frac{\partial}{\partial t} \left(\frac{1}{a} \right) v = \frac{\partial^2 v}{\partial x^2}.$$

If we put

$$1/a = 2 - \psi\{(1-t)^{-1/2}[1 - |x|(1-t)^{-1/2}]\}/\ln(1-t-x^2),$$

then, literally repeating the reasoning for the preceding equation, we arrive at the conclusion that, for some smooth boundary conditions, the solution $u(x, t)$ of equation (10) tends to infinity for $x = 0$, $t \rightarrow 1$.

The self-adjoint equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x} \left[a(x, t) \frac{\partial u}{\partial x} \right]$$

is reduced to the preceding one by the substitution $\partial u/\partial x = w$ and differentiation with respect to x . Consequently, for some boundary conditions,

$$\frac{\partial u}{\partial x}(0, t) \xrightarrow{t \rightarrow 1} \infty.$$

In conclusion we present some known positive results concerning the existence of a classical solution of a parabolic equation and its a priori estimates. In the paper ⁽²⁾ the existence of a fundamental solution was proved, and together with it also that of a classical solution for parabolic equations and systems, under the condition that the coefficients of the highest derivatives are continuous and satisfy the Dini condition with respect to the spatial variables. The example

constructed shows that this result is almost sharp. In equation (2) the modulus of continuity has order $(\ln h)^{-1}$, while functions with modulus of continuity $(\ln h)^{-\alpha}$, $\alpha > 1$, already satisfy the Dini condition. The absence of a fundamental solution for a parabolic equation with discontinuous coefficients was shown in (3).

For the one-dimensional equation (2), in the work (4) a priori estimates of the solution in Hölder norms were proved, depending only on the maximum and minimum of the function $a(x, t)$, and the Hölder exponent α may be any number less than one. For a self-adjoint parabolic equation in the multidimensional case, an estimate of the solution in a Hölder norm is proved in the paper (5), but the exponent $\alpha < 1$ depends on the dimension and on the eigenvalues of the matrix $\|a_{ij}\|$. The example given above shows that an estimate with $\alpha = 1$ is impossible even if a sufficiently weak modulus of continuity of the coefficients is included in the estimate. It remains unclear whether one can obtain an estimate for any $\alpha < 1$.

If in the equation

$$\frac{\partial u}{\partial t} = \sum_{i,j=1}^n \frac{\partial^2}{\partial x_i \partial x_j} (a_{ij}(x, t)u) \quad (11)$$

the coefficients a_{ij} have first derivatives with respect to the spatial variables, bounded in $L_q(E_n)$ for $q > n$, then a priori estimates in Hölder norms follow from the results of the paper (6). The author is not aware of any estimates of solutions of equation (11) in uniform norms under weaker restrictions on the coefficients. The example of equation (10) indicates possible limits of positive results in this case as well.

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Note: Figure translations are in progress. See original paper for figures.

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