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Abstract

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ON SOME NEW RESULTS IN THE THEORY OF RESOLVENTS OF HERMITIAN OPERATORS

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1. Let $\mathfrak{H}$ be a Hilbert space; A a certain simple closed Hermitian operator acting in $\mathfrak{H}$, with domain of definition $\mathfrak{D}(A)$ dense in $\mathfrak{H}$, and having equal defect numbers
$$n_+(A) = n_-(A)(= n(A)).$$
Put
$$\mathfrak{M}_z = (A - zI)\mathfrak{D}(A)$$
(so that $n(A) = \dim(\mathfrak{H} \ominus \mathfrak{M}_z)$ for $\operatorname{Im} z \neq 0$).

In 1943, independently of one another, M. A. Naimark ([1]) and M. G. Krein ([2]) obtained a description of all generalized resolvents of a Hermitian operator with $n(A) = 1$. Subsequently these results were generalized to the case of arbitrary natural $n(A)$ ([3]) and, further, in works of A. V. Shtraus ([4,5]) to the case of any equal or unequal $n_\pm(A) \leq \infty$.

It should be pointed out, however, that only in ([2]) was a description of generalized resolvents given by means of the **resolvent matrix**–a description adapted to the aims of the theory of entire Hermitian operators ([6,7]) and of the general theory of representations of Hermitian operators ([2,7,8]). This result was generalized by M. G. Krein to the case of arbitrary natural $n(A)$, but was not published.

In the present communication we set forth the basic propositions of the theory of the **resolvent $\mathfrak{L}$-matrix** in the general case $n(A) \leq \infty$. The systematic use of the projector-function $\mathfrak{P}(z)$ and of the adjoint operator-function $Q(z)$, generated by the operator A and the representation subspace $\mathfrak{L}$ (in ([7]) it was denoted by M), made it possible to write, in a new compact form, the cumbersome relations previously established only for special cases ($n(A) = 1$ or $n(A) < \infty$); at the same time more natural and transparent proofs were obtained.

2. Let $\mathfrak{L}(\subset \mathfrak{H})$ be a certain subspace. We shall call a point z $\mathfrak{L}$-regular for A if $\mathfrak{M}_z$ is closed, $\mathfrak{M}_z \cap \mathfrak{L} = \{0\}$, and, moreover,

$$\mathfrak{H} = \mathfrak{M}_z \dotplus \mathfrak{L}$$

($\dotplus$ is the sign of a direct sum). It is easily shown that the set $\rho(A;\mathfrak{L})$ of all $\mathfrak{L}$-regular points for the operator A is open.

If $z \in \rho(A;\mathfrak{L})$, then by $\mathfrak{P}(z)$ we denote the operator of projection of $\mathfrak{H}$ onto $\mathfrak{L}$ parallel to $\mathfrak{M}_z$; thus, for any $f \in \mathfrak{H}$ one has: 1) $\mathfrak{P}(z)f \in \mathfrak{L}$; 2) $f - \mathfrak{P}(z)f \in \mathfrak{M}_z$. For such z one may further define the operator $Q(z)$ by the equality

$$Q(z)f = P_{\mathfrak{L}}(A - zI)^{-1}(f - \mathfrak{P}(z)f), \tag{1}$$

where $P_{\mathfrak{L}}$ is the operator of orthogonal projection of $\mathfrak{H}$ onto $\mathfrak{L}$.

It is shown without particular difficulty that $\mathfrak{P}(z)$ and $Q(z)$ are piecewise holomorphic functions on $\rho(A;\mathfrak{L})$ (holomorphic operator-functions in any domain contained in $\rho(A;\mathfrak{L})$).

Denote by $\rho_s(A;\mathfrak{L})$ the intersection of $\rho(A;\mathfrak{L})$ with its mirror reflection with respect to the real axis:

$$\rho_s(A;\mathfrak{L}) = \rho(A;\mathfrak{L}) \cap \overline{\rho(A;\mathfrak{L})}.$$

Thus, $z \in \rho_s(A;\mathfrak{L})$ if $z, \bar{z} \in \rho(A;\mathfrak{L})$. Everywhere in what follows, unless the contrary is stipulated, we assume that A and $\mathfrak{L}$ are such that the set $\rho_s(A;\mathfrak{L})$ is nonempty and hence decomposes into a countable sum of open domains-

properties. By $\mathfrak{B}_1(\mathfrak{L})$ we shall denote the Banach algebra of all linear bounded operators acting in $\mathfrak{L}$, and by $\mathfrak{B}_2(\mathfrak{L})$ the set of all matrices of order two with entries from $\mathfrak{B}_1(\mathfrak{L})$. Obviously, $\mathfrak{B}_2(\mathfrak{L})$ may also be regarded as a certain noncommutative symmetric algebra*.

Put**

$$\mathfrak{y} = i\mathfrak{y}_0 = i\begin{pmatrix} 0 & -I_{\mathfrak{L}} \\ I_{\mathfrak{L}} & 0 \end{pmatrix}, \qquad G_z = \begin{pmatrix} \mathcal{P}(z) \\ Q(z) \end{pmatrix} \quad (z \in \rho(A;\mathfrak{L})). \tag{2}$$

Theorem 1. There exists a matrix-function $W(z) = \|\mathcal{W}_{jk}(z)\|_1^2$, piecewise holomorphic in $\rho_s(A;\mathfrak{L})$, with values in $\mathfrak{B}_2(\mathfrak{L})$, invertible in $\mathfrak{B}_2(\mathfrak{L})$, such that for $z, \zeta \in \rho_s(A, \mathfrak{L})$:

$$W(z)\mathfrak{y}W^*(\zeta) = \mathfrak{y} + \frac{1}{i}(z - \bar{\zeta})G(z)G^*(\zeta)\,\big|_{\mathfrak{L}} =$$

$$= i\begin{pmatrix} (z-\bar{\zeta})\mathcal{P}(z)\mathcal{P}^*(\zeta)\,\big|_{\mathfrak{L}} & I_{\mathfrak{L}} + (z-\bar{\zeta})\mathcal{P}(z)Q^*(\zeta)\,\big|_{\mathfrak{L}} \\ -I_{\mathfrak{L}} + (z-\bar{\zeta})Q(z)\mathcal{P}^*(\zeta)\,\big|_{\mathfrak{L}} & (z-\bar{\zeta})Q(z)Q^*(\zeta)\,\big|_{\mathfrak{L}} \end{pmatrix}. \tag{3}$$

By this identity the matrix-function $W(z)$ is determined uniquely up to the transformation $W(z) \to W(z)\mathcal{E}$, where $\mathcal{E}$ is a constant $\mathfrak{y}$-unitary matrix from $\mathfrak{B}_2(\mathfrak{L})$.

Let us explain that a matrix $\mathcal{E} \in \mathfrak{B}_2(\mathfrak{L})$ is called $\mathfrak{y}$-unitary if it is invertible (there exists $\mathcal{E}^{-1} \in \mathfrak{B}_2(\mathfrak{L})$) and $\mathcal{E}^*\mathfrak{y}\mathcal{E} = \mathfrak{y}$ (and hence also $\mathcal{E}\mathfrak{y}\mathcal{E}^* = \mathfrak{y}$).

It follows from (3) that the matrix $W^*(z)$, for $\operatorname{Im} z > 0$ ($\operatorname{Im} z < 0$), is $\mathfrak{y}$-expanding ($\mathfrak{y}$-contracting); for real $z = a \in \rho(A, \mathfrak{L})$ the matrix $W^*(a)$, and hence also $W(a)$, are $\mathfrak{y}$-unitary.

If the operator A has a real point a of regular type, then the matrix-function $W(z)$ can be expressed explicitly through the operator-functions $\mathcal{P}(z)$ and $Q(z)$. It turns out that in this case the matrix-function $W(z)$ can be chosen so that $W(a)$ is the identity matrix I_2 from $\mathfrak{B}_2(\mathfrak{L})$. Putting $\zeta = a$ in (3), we find $W(z)$. Thus, the following holds.

Theorem 2. If some real a is a point of regular type for the operator A, then the resolvent $\mathfrak{L}$-matrix-function $W(z)$ of the operator A can be obtained by the formula

$$W(z) = I_2 + (z - a)G(z)G^*(a)\mathfrak{y}_0. \qquad (4)$$

3. We now dwell on the particular case when $n(A) = 1$, and hence $\mathfrak{L}$ is a one-dimensional subspace: $\mathfrak{L} = \{\xi u\}$ ($\|u\| = 1$, ξ a scalar). Let $\{\varphi_j\}_1^\infty$ be some orthonormal basis of the space $\mathfrak{H}$. In this case, if the set $\rho_s(A; \mathfrak{L})$ is nonempty, then the nonreal points z not belonging to it form an isolated set inside the upper and lower half-planes (see ([2],[7])).

By virtue of the one-dimensionality of $\mathfrak{L}$, for every $f \in \mathfrak{H}$ we shall have $\mathcal{P}_z(z)f = f(z)u$, where $f(z)$ is a certain piecewise holomorphic function on $\rho_s(A; \mathfrak{L})$. Define on $\rho_s(A; \mathfrak{L})$ the functions $\varphi_j(z)$ and $\psi_j(z)$ ($j = 1, 2, \ldots$), putting $\mathcal{P}(z)\varphi_j = \varphi_j(z)u$, $Q(z)\psi_j = \psi_j(z)u$, or, equivalently, $\varphi_j(z) = (\mathcal{P}(z)\varphi_j, u)$, $\psi_j(z) = ((A - zI)^{-1}(\varphi_j - \varphi_j(z)u), u)$, ($j = 1, 2, \ldots$).

In the case under consideration the operators $\mathcal{W}_{jk}(z)$ ($j, k = 1, 2$) from $\mathfrak{B}(\mathfrak{L})$ may be regarded as operators of multiplication by scalar functions,

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* If $W \in \mathfrak{B}_2(\mathfrak{L})$, then by W^* we shall denote the matrix obtained from the matrix W by transposing it and replacing all its entries by the adjoint operators.

** $I_{\mathfrak{L}}$ is the identity operator in $\mathfrak{L}$. By $C|_{\mathfrak{L}}$ is denoted the restriction of the operator C to $\mathfrak{L}$.

coinciding respectively with the functions $w_{jk}(z) = (\mathfrak{W}_{jk}(z)u, u)$ ($j, k = 1, 2$).

Let us note that

$$(\mathfrak{P}(z)\mathfrak{P}^*(\zeta)u,u) = (\mathfrak{P}^*(\zeta)u,\mathfrak{P}^*(z)u) = \sum_{j=1}^{\infty}(\mathfrak{P}^*(\zeta)u,\varphi_j)(\varphi_j,\mathfrak{P}^*(z)u) =$$

$$= \sum_{j=1}^{\infty}(u,\mathfrak{P}(\zeta)\varphi_j)(\mathfrak{P}(z)\varphi_j,u) = \sum_{j=1}^{\infty}\varphi_j(z)\overline{\varphi_j(\zeta)} \tag{5}$$

and that, in an analogous way, the functions $(Q(z)Q^*(\zeta)u,u)$ and $(\mathfrak{P}(z)Q^*(\zeta)u,u)$ can be expanded in series. Therefore, if one applies the operators expressed by the left- and right-hand sides of equality (3) to the vector u and then takes the scalar product of the obtained vectors with u, this gives us the following scalar identities:

$$q_0(z)\overline{q_1(\zeta)} - q_1(z)\overline{q_0(\zeta)} = (z-\bar{\zeta})\sum_{j=1}^{\infty}\varphi_j(z)\overline{\varphi_j(\zeta)},$$

$$p_0(z)\overline{p_1(\zeta)} - p_1(z)\overline{p_0(\zeta)} = \sum_{j=1}^{\infty}\psi_j(z)\overline{\psi_j(\zeta)}, \tag{6}$$

$$q_0(z)\overline{p_1(\zeta)} - q_1(z)\overline{p_0(\zeta)} = 1-(z-\bar{\zeta})\sum_{j=1}^{\infty}\varphi_j(z)\overline{\psi_j(\zeta)},$$

where we have put

$$p_1(z) = (\mathfrak{W}_{11}(z)u,u), \qquad p_0(z) = (\mathfrak{W}_{12}(z)u,u),$$

$$q_1(z) = (\mathfrak{W}_{21}(z)u,u), \qquad q_0(z) = (\mathfrak{W}_{22}(z)u,u).$$

These identities were first reported in 1943 in the note ([2]). They also have meaning for the case of finite-dimensional $\mathfrak{H}$, and in this case lead to relations of the theory of orthogonal polynomials known under the name of Christoffel identities.

Thus, identity (3) should be regarded as a further generalization of the Christoffel identities. It contains, as special cases, various identities that have been used in the classical power moment problem and in its various matrix and operator generalizations ([9], [10]), in the problem of continuation of Hermitian-positive scalar, matrix, and operator functions ([7], [11], [12]), and others.

4. Let us also note some important details for the case when the Hermitian operator A is real (with respect to a certain involution $f \to \bar{f}$ of complex conjugation given in $\mathfrak{H}$). In this case one always has $n_+(A) = n_-(A)$.

If now one chooses as $\mathfrak{L}$ a subspace invariant with respect to the involution (i.e. having a real orthonormal basis), then we shall have $\rho_s(A;\mathfrak{L}) = \rho(A;\mathfrak{L})$.

Theorem 3. *If the Hermitian operator A is real and $\mathfrak{L} = \overline{\mathfrak{L}}$, then the resolvent $\mathfrak{L}$-matrix-function can be chosen so that: 1) $W(\bar{z}) = \overline{W(z)}$ $(z \in \rho(A;\mathfrak{L}))$. With such a choice it will be symmetric, i.e. 2) $W^t(z)\mathfrak{J}W(z) = \mathfrak{J}$ $(z \in \rho(A;\mathfrak{L}))$.*

If, moreover, the operator A has a real point $a \in \rho(A;\mathfrak{L})$, then the function $W(z)$ with properties 1) and 2) can be obtained by formula (4).

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* If $C \in \mathfrak{B}_1(\mathfrak{H})$, then by $\overline{C}$ is denoted the operator defined by the equality $\overline{C}f = \overline{C\overline{f}}$ $(f \in \mathfrak{H})$; by C^t is denoted the transposed operator, defined by the equality: $C^t = \overline{C}^*$ $(= (\overline{C})^*)$. After this, the symbols $\overline{W}$ and W^t are naturally defined for every $W \in \mathfrak{B}_2(\mathfrak{H})$.

5. A monotone operator-function $\sigma(\lambda) = \sigma(\lambda - 0)$ $(-\infty < \lambda < \infty)$, whose values are bounded self-adjoint operators acting in L, will be called an L-spectral function of the operator A, if it is representable in the form $\sigma(\lambda) = P_{\mathfrak{L}}E(\lambda)P_{\mathfrak{L}}$ $(-\infty < \lambda < \infty)$, where $E(\lambda)$ is some generalized spectral function of the Hermitian operator A; an $\mathfrak{L}$-spectral function is called orthogonal if $E(\lambda)$ is an orthogonal spectral function of the operator A.

Define the operator-functions $\mathcal{A}(z)$, $\mathcal{B}(z)$, $\mathcal{C}(z)$, $\mathcal{D}(z)$ by the equality

$$\begin{pmatrix} \mathcal{W}_{11}(z) & \mathcal{W}_{12}(z) \\ \mathcal{W}_{21}(z) & \mathcal{W}_{22}(z) \end{pmatrix} \cdot \begin{pmatrix} I_{\mathfrak{L}} & I_{\mathfrak{L}} \\ iI_{\mathfrak{L}} & -iI_{\mathfrak{L}} \end{pmatrix} = \begin{pmatrix} \mathcal{A}(z) & \mathcal{B}(z) \\ \mathcal{C}(z) & \mathcal{D}(z) \end{pmatrix} \qquad (z \in \rho_s(A;\mathfrak{L})) \quad (7)$$

and denote by $C(\mathfrak{L})$ the totality of all operator-functions $\Omega(z)$, holomorphic inside the upper half-plane, whose values are contraction operators in $\mathfrak{L}$ $(\|\Omega(z)\| \le 1, \operatorname{Im} z > 0)$.

The following theorem gives a complete description of the set $S(A;\mathfrak{L})$ of all $\mathfrak{L}$-spectral functions of the operator A.

Theorem 4. *Between the sets $S(A;\mathfrak{L})$ and $C(\mathfrak{L})$ there can be established a one-to-one correspondence $\sigma(\lambda) \leftrightarrow \Omega(z)$, by virtue of which*

$$[\mathcal{A}(z)\Omega(z) + \mathcal{B}(z)][\mathcal{C}(z)\Omega(z) + \mathcal{D}(z)]^{-1} = \int_{-\infty}^{\infty} \frac{d\sigma(\lambda)}{\lambda - z} \qquad (\operatorname{Im} z > 0). \quad (8)$$

Thus, the theorem contains the assertion that for any function $\Omega(z) \in C(\mathfrak{L})$ the left-hand side of (8) has meaning on $\rho_s(A;\mathfrak{L})$ and, moreover, has an analytic continuation to the whole half-plane $\operatorname{Im} z > 0$. If $\|\Omega(z)\| < 1$ for $\operatorname{Im} z > 0$, then, putting $T(z) = i(I_{\mathfrak{L}} + \Omega(z))(I_{\mathfrak{L}} - \Omega(z))^{-1}$, it will be possible to rewrite the left-hand side of (8) in the form

$$[\mathcal{W}_{11}(z)T(z) + \mathcal{W}_{12}(z)][\mathcal{W}_{21}(z)T(z) + \mathcal{W}_{22}(z)].$$

It is easy to see that the operator-function $T(z)$, holomorphic inside the upper half-plane and with values in $\mathfrak{B}_1(\mathfrak{L})$, is characterized by the property $\operatorname{Im}(T(z)f, f) > 0$ for $f \in \mathfrak{L}$, $f \neq 0$. Taking this circumstance into account, Theorem 4 should be regarded as a far-reaching generalization of Theorem 7 from ([2]). In deriving Theorem 4, the works ([3],[4],[13]) were used.

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Note: Figure translations are in progress. See original paper for figures.