ON FUNCTIONAL EQUATIONS OF DIRICHLET FUNCTIONS

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Abstract

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A. F. LAVRIK

ON FUNCTIONAL EQUATIONS OF DIRICHLET FUNCTIONS

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Let k be an integer ≥ 1; let $A, B, \delta, \alpha_1, \dots, \alpha_k$ be arbitrary positive real numbers; let $\lambda; \beta_1, \dots, \beta_k$ be complex; let $\Gamma(z)$ be the gamma function;

$$\Gamma_k(z) = \prod_{\nu=1}^{k} \Gamma(\alpha_\nu z + \beta_\nu);$$

suppose that in the complex s-plane, except possibly for a finite number of its points, we have

$$A^s \Gamma_k(s) \varphi(s) = \lambda B^{\delta - s} \Gamma_k(\delta - s) \psi(\delta - s), \tag{1}$$

where $\varphi(s)$, $\psi(s)$ are functions defined for $\operatorname{Re} s > \delta$ by absolutely convergent Dirichlet series

$$\varphi(s) = \sum_{n=1}^{\infty} \frac{a_n}{\lambda_n^s}, \qquad \psi(s) = \sum_{n=1}^{\infty} \frac{b_n}{\xi_n^s}. \tag{2}$$

Then (1) is the most familiar form of the functional equation of Dirichlet functions.

In works ([1-4]), for $\varphi(s) = L(s, \chi)$, so-called shortened functional equations, very useful in applications, were obtained in the critical strip. In the present paper we consider the question of such equations in general form and for arbitrary functions (2). In this, the requirement of the equation (1) is necessary and sufficient.

We introduce additional definitions and notation. Let $\alpha = \alpha_1 + \dots + \alpha_k$; $\beta = \beta_1 + \dots + \beta_k$; for a given s choose arbitrary

$$\gamma_0 < \min(0; \operatorname{Re} s); \qquad \sigma_0 > \max(\delta; \delta + \operatorname{Re} s; -\operatorname{Re} \beta_\nu / \alpha_\nu),$$

$$\nu = 1, \dots, k; \qquad \gamma = -\gamma_0 + \operatorname{Re} s; \qquad \sigma = \sigma_0 - \operatorname{Re} s.$$

Let Ω denote the strip of the ω-plane with the condition $\gamma_0 \le \operatorname{Re}\omega \le \sigma_0$;

$$\Gamma_{k\Delta}(\omega; x, \Phi) = \frac{1}{2\pi i} \int_{(\Delta)} \Gamma_k(z+\omega) x^z \Phi(\pm z)\, dz,$$

where, depending on whether $\omega = s$ or $\omega = \delta - s$, Δ is taken equal to σ and $\Phi(z)$, or else $\Delta = \gamma$ and $\Phi(-z)$, and the integration is carried out along the line $\operatorname{Re} z = \Delta$ in the positive direction.

Main Lemma. Let the functions $\varphi(s)$, $\psi(s)$ from (2) satisfy equation (1), and let $F_k(s) = \Gamma_k(s)\varphi(s)$ be a regular function in the domain Ω, with the possible exception of a finite number of pole-points. Then for every s distinct from the poles of $F_k(s)$, we have

$$A^s \sum_{n=1}^{\infty} \frac{a_n}{\lambda_n^s} \Gamma_{k\sigma}\left(s; \frac{A}{\lambda_n}, \Phi\right) - \lambda B^{\delta-s} \sum_{n=1}^{\infty} \frac{b_n}{\xi_n^{\delta-s}} \Gamma_{k\gamma}\left(\delta - s; \frac{B}{\xi_n}, \Phi\right) =$$

$$= \sum_{\omega\in\Omega} \operatorname{res}\{A^\omega F_k(\omega)\Phi(\omega - s)\},$$

where the series converge absolutely for any regular value, except, possibly, for a finite number of pole-points, in Ω, of the function $\Phi(\omega)$, such that, for $\gamma_0 \le y \le \sigma_0$, $|t| = T \to \infty$,

$$|\Phi(y+iT)| \ll \exp\left(\frac{\pi\alpha}{2} T\right) T^{-l-1},$$

$$l = \frac{a(\sigma - y)(\delta - 2\gamma)}{\sigma - \gamma} + ay + \operatorname{Re}\beta - \frac{k}{2}.$$

The requirements imposed here on the function Φ cannot be substantially weakened in the sense that the $\pi/2$ under the exponential sign cannot be replaced by any other constant, however much larger.

We next state, in the form of theorems, some corollaries that follow from this lemma.

Theorem 1. If the function $\Phi(\omega)$ has a simple pole at the point $\omega = 0$, then

$$\varphi(s)\Gamma_k(s) \lim_{z\to 0} z\Phi(z) =$$

$$= \sum_{n=1}^{\infty} \frac{a_n}{\lambda_n^s} \Gamma_{k\sigma} \left(s; \frac{A}{\lambda_n}, \Phi \right) - \lambda \frac{B^{\delta - s}}{A^s} \sum_{n=1}^{\infty} \frac{b_n}{\xi_n^{\delta - s}} \Gamma_{k\gamma} \left(\delta - s; \frac{B}{\xi_n}, \Phi \right) +$$

$$+ \sum_{\substack{\omega \neq s \\ \omega \in \Omega}} \operatorname{res} \{ A^{\omega - s} \Gamma_k(\omega) \varphi(\omega) \Phi(\omega - s) \},$$

where the last sum extends over all poles from the domain Ω distinct from the pole at the point $\omega = s$, provided that they exist. Otherwise this sum is taken to be zero.

As a concretization of this theorem one obtains shortened functional equations, by means of which one can find estimates, uniform in the main parameters, for Dirichlet functions in the critical strip.

In particular, let $\gamma_\nu = \alpha\beta_\nu/\alpha_\nu$, $\nu = 1, \dots, k$,

$$\Gamma_k(\omega; x) = \frac{\alpha^k}{\alpha_1 \dots \alpha_k} x^{\alpha\omega + \beta} \int_{\substack{\xi_1 \dots \xi_k \geq 1 \\ \xi_j > 0}} \cdots \int \prod_{\nu=1}^{k} \exp[-x \xi_\nu^{\alpha/\alpha_\nu}] \xi_\nu^{\alpha\omega + \gamma_\nu - 1} \, d\xi_1 \dots d\xi_k. \tag{3}$$

Then the following holds.

Theorem 2. For all z and s such that

$$|\arg z| < \pi/2; \qquad 0 \leq \operatorname{Re} s \leq \delta,$$

$$\varphi(s)\Gamma_k(s) = \sum_{n=1}^{\infty} \frac{a_n}{\lambda_n^s} \Gamma_k \left(s; \lambda_n^{1/\alpha} A^{-1/\alpha} z \right) +$$

$$+ \lambda \frac{B^{\delta - s}}{A^s} \sum_{n=1}^{\infty} \frac{b_n}{\xi_n^{\delta - s}} \Gamma_k \left(\delta - s; \xi_n^{1/\alpha} B^{-1/\alpha} z^{-1} \right) + \sum_{\omega \neq s} \operatorname{res} \left\{ \frac{\Gamma_k(\omega)\varphi(\omega)}{\omega - s} \left(\frac{z^\alpha}{A} \right)^{s - \omega} \right\}.$$

Let us make several remarks on the substance of this theorem. For $k = 1$, $\Gamma_k(\omega; x)$ is the ordinary incomplete gamma-function. If the integral analogous to (3), but over the domain $\xi_1 \dots \xi_k \leq 1$, $\xi_j > 0$, is denoted by $\gamma_k(\omega; x)$, then we shall have

$$\Gamma_k(\omega) = \Gamma_k(\omega; x) + \gamma_k(\omega; x).$$

Consequently, $\Gamma_k(\omega; x)$ is naturally regarded as an incomplete k-dimensional gamma-function. Its behavior for $k > 1$ is essentially the same as in the case

$k = 1$. Namely, $\Gamma_k(\omega; x)$ is the product of $\exp(-\operatorname{Re} x)$ and a power function of $|x|$ and $|\operatorname{Im} s|$, so that $\varphi(s)\Gamma_k(s)$ is determined mainly by the initial segments of the series, until a sharp drop occurs owing to the influence of the factor $\exp(-\operatorname{Re} x)$.

If one takes z such that

$$\arg z = \left(\frac{\pi}{2} - \frac{1}{|\operatorname{Im} s| + 1}\right) \operatorname{sgn} \operatorname{Im} s,$$

then from each term of these segments there will also be separated a factor that suppresses the exponential growth $[\Gamma_k(s)]^{-1}$ as $|\operatorname{Im} s| \to \infty$.

In other words, equation (4) combines within itself both the exact equation (1) and the approximate equation.

In conclusion we note one more consequence of the lemma.

Theorem 3. *If $\Gamma_k(s)\varphi(s)$ is an entire function, then for the existence of the functional equation (1) it is necessary and sufficient that*

$$\sum_{n=1}^{\infty} a_n \Theta\left(\frac{A}{\lambda_n}\right) = \lambda \sum_{n=1}^{\infty} b_n \Theta\left(\frac{B}{\xi_n}\right),$$

where

$$\Theta(x) = \frac{1}{2\pi i} \int_{(\sigma)} \Gamma_k(z) x^z \, dz,$$

and the integration is taken along a vertical line to the right of which $\Gamma_k(z)$ has no poles.

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Note: Figure translations are in progress. See original paper for figures.