ELLIPTIC BOUNDARY-VALUE PROBLEMS WITH A PARAMETER ONLY IN THE BOUNDARY CONDITIONS

MATHEMATICS

1966

SovietRxiv

Abstract

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UDC 517.43

MATHEMATICS

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ELLIPTIC BOUNDARY-VALUE PROBLEMS WITH A PARAMETER ONLY IN THE BOUNDARY CONDITIONS

(Presented by Academician I. G. Petrovskii, 10 I 1966)

In the present paper we consider elliptic boundary-value problems in which the equations do not depend on the parameter, while the boundary conditions do. The case in which both the equations and the boundary conditions depend on the parameter was considered by M. S. Agranovich and M. I. Vishik ([1]). They found a criterion under which the problem has a solution, and moreover a unique one, for sufficiently large values of the parameter. We have extended the indicated result to the case in which only the boundary conditions depend on the parameter. The author was led to the need to consider the latter case while studying the question of an equivalent regularization of elliptic boundary-value problems by means of potentials. Along the way, in the present paper a special representation is obtained for the normal derivatives of solutions of an elliptic system in terms of Dirichlet data on the boundary.

1. Let Ω be a certain bounded domain in R^n with infinitely smooth boundary Γ; the smoothness conditions imposed on Γ and on the functions occurring below can be weakened, but we shall not do this. Consider in Ω a system of differential equations, each of order $2m$,

$$\sum_{k=1}^{s} A_{jk}\left(x, -i\frac{\partial}{\partial x}\right) u_k = 0 \qquad j = 1, 2, \dots, s, \tag{1}$$

with boundary conditions

$$\left[\sum_{k=1}^{m} B_{jk}\left(x, -i\frac{\partial}{\partial x}, q\right) u_k\right]_{x\in\Gamma} = f_j(x), \qquad j = 1, 2, \dots, ms, \tag{2}$$

where

$$A_{jk}(x,\xi) = \sum_{|\alpha|=1}^{2m} a_{jk,\alpha}(x)\xi^{\alpha};$$

$$B_{jk}(x,\xi,q) = \sum_{|\alpha|+\beta\le m_j} a_{jk,\alpha\beta}(x)q^{\beta}\xi^{\alpha};$$

$x = (x_1,\dots,x_n)$, $\xi = (\xi_1,\dots,\xi_n)$ are points of R^n, $\xi^\alpha = \xi_1^{\alpha_1}\xi_2^{\alpha_2}\cdots\xi_n^{\alpha_n}$, $|\alpha| = \alpha_1 + \dots + \alpha_n$; q is a parameter. The coefficients $a_{jk,\alpha}(x)$, $b_{jk,\alpha\beta}(x)$ and the functions $f_j(x)$ are assumed infinitely smooth. By $A_{jk}^{(0)}$ and $B_{jk}^{(0)}$ we shall denote the principal parts of the corresponding operators, selecting in the first case the terms with the highest derivatives, and in the second the highest terms with respect to the sum of the orders of the derivatives and the powers of the parameter q. In abbreviated form, system (1) and boundary conditions (2) will be written as

$$A(x,-i\partial/\partial x)u = 0, \qquad B(x,-i\partial/\partial x,q)u = f(x).$$

We shall seek the solution in the classes $w_2^{(l)}(\overline{\Omega})$, $l \ge \max(2m, m_j)$. Instead of these spaces one could take $C_l(\overline{\Omega})$.

2. We make the following assumptions.

Assumption I. The system (1) is elliptic in the sense of Petrovskii.

Assumption II. The Dirichlet problem for the system (1) in the domain $\overline{\Omega}$ ($\overline{\Omega} = \Omega + \Gamma$)

$$\partial^p u_k/\partial\nu^p = \varphi_k^{(p)}(x) \tag{3}$$

($k = 1,2,\dots,s$; $p = 0,1,\dots,m-1$; $\partial/\partial\nu$ is differentiation along the inward normal)

is normally solvable and uniquely solvable.

Assumption III. Let $K_\eta(x_0)$ be the ball of radius η with center at the point x_0. We require that there exist a sufficiently small $\eta > 0$, the same for all $x_0 \in \Gamma$, such that for every $x_0 \in \Gamma$ and for every sufficiently small domain $\overline{\Omega}_{x_0}$, $\overline{\Omega} \cap K_\eta(x_0) \subset \overline{\Omega}_{x_0} \subset \overline{\Omega}$, the Dirichlet problem for the system $A^{(0)}(x_0,-i\,\partial/\partial x)u = 0$ be normally solvable and uniquely solvable.

For the condition of normal solvability of elliptic boundary-value problems (the Shapiro–Lopatinskii condition), see the papers ([1,3–6]). In the case of the Dirichlet problem this condition is explicitly written out in ([5,6]).

Assumption III may be replaced by the following:

Assumption III′. The system (1) has in the domain $\overline{\Omega}$ a fundamental solution "as a whole."

As Yu. I. Lyubich showed ([7]), for this it is sufficient that the Cauchy problem be uniquely solvable "locally." If Assumptions II and III′ are fulfilled, then there exists the Green matrix of the Dirichlet problem for the system (1) in the domain $\overline{\Omega}$.

3. We formulate a theorem on the representation of the normal derivatives on Γ of solutions of the system (1) in terms of the Dirichlet data. Introduce the notation

$$\partial^p u_k/\partial \nu^p = g_{kp}, \qquad k = 1, 2, \dots, s; \quad p = 0, 1, \dots, m-1. \tag{4}$$

For short, we shall denote the matrix $\{g_{kp}\}$ by g. If Assumptions I and II are valid, then any solution of the system (1) is representable in the form

$$u_j(x) = \sum_{k=1}^{s} \sum_{p=0}^{m-1} C_{jkp}(x, g_{kp}), \tag{5}$$

where C_{jkp} are bounded operators from $W_2^{(l-p-1/2)}(\Gamma)$ to $W_2^{(l)}(\overline{\Omega})$ (see ([1,3–5])).

Let $x_0 \in \Gamma$. Draw at the point x_0 the tangent hyperplane to Γ and "freeze" at this point all the coefficients of the equations of the system (1). We shall assume that the system has already been written in a local coordinate system at the point x_0, i.e. the axis Ox_n is directed along the normal, while all the remaining axes lie in the tangent hyperplane. Consider the boundary-value problem in the half-space $x_n > 0$ (R^n_+):

$$A^{(0)}(x_0, -i\partial/\partial x)v = 0, \qquad \partial^p v_k/\partial x_n^p = \tilde{g}_{kp}. \tag{6}$$

By virtue of the assumption on normal solvability of the Dirichlet problem for the original system, this boundary-value problem is solvable; the solution can be found by means of the Fourier transform. Put

$$\left.\frac{\partial^r v_j}{\partial x_n^r}\right|_{x_n=+0} = \sum_{k=1}^{s} \sum_{r=0}^{m-1} D^{x_0}_{jkpr}(x', \tilde{g}_{kp}). \tag{7}$$

For $r \le m-1$ we have

$$D^{x_0}_{jkpr}(x', \tilde{g}_{kp}) = \delta_{jk}\delta_{pr}\tilde{g}_{kp}(x').$$

For $r \geq m$, $D^{x_0}_{jkpr}$ is a singular integro-differential operator (s.i.-d. operator) of order $r-p$ (see ([2, 3, 5])) with symbol independent of the point $x' = (x_1, \ldots, x_{n-1})$.

Theorem 1. Suppose that assumptions I, II, and III (or III′) are satisfied. Then for $r \geqslant m$ we have

$$\left(\frac{\partial C_{jkp}(x, g_{kp})}{\partial \nu^r}\right)_{x_0} = E^{(0)}_{jkpr}(x_0, g_{kp}) + E^{(1)}_{jkpr}(x, g_{kp}) + E^{(2)}_{jkpr}(x_0, g_{kp}), \quad (8)$$

where $E^{(0)}_{jkpr}$ is a singular integro-differential operator on the manifold Γ of order $r-p$, whose symbol in a local coordinate system coincides with the symbol of the operator $D^{(1)}_{jkpr}$; $E^{(1)}_{jkpr}$ is a singular integro-differential operator on Γ of order $r-p$, whose norm can be made arbitrarily small; $E^{(2)}_{jkpr}$ is a singular integro-differential operator on Γ of order lower than $r-p$. For $r \leqslant m-1$ we have*

$$\left(\frac{\partial C_{jkp}(x, g_{kp})}{\partial \nu^r}\right)_{x_0} = \delta_{jk}\delta_{pr} g_{kp}(x_0). \quad (9)$$

4. Let us return to the original boundary-value problem. Obviously, the boundary conditions can be reduced to the form:

$$\left[\sum_{k=1}^{s}\sum_{r=0}^{m_j} \overline{B}_{jkr}\left(x', -i\frac{\partial}{\partial x'}, q\right)\frac{\partial u_k}{\partial \nu^r}\right]_{x=x'\in\Gamma} = f_j(x'); \quad (10)$$

$\overline{B}_{jkr}$ is a purely tangential differential operator of order $m_j - r$; if $m_j - r < 0$, then $\overline{B}_{jkr} \equiv 0$.

We shall seek the solution of the original boundary-value problem in the form (5). Substituting (5) into the boundary conditions, we obtain a certain system of integro-differential equations for g_{kp} on the closed manifold Γ. This system depends on a parameter, and the results of (1) are applicable to it. In abbreviated form we shall write this system as

$$\widehat{B}\left(x', -i\frac{\partial}{\partial x'}, q\right) g = f(x'). \quad (11)$$

The highest derivatives of g_{kp} in the j-th equation of system (11) have order $m_j - p$; if $m_j - p < 0$, then the corresponding term is absent. Consequently, system (11) is normal in the sense of Douglis–Nirenberg (see (4)).

Normal solvability of the original boundary-value problem is equivalent to ellipticity of system (11). If, however, the principal terms are selected not according to derivatives, but according to the sum of the degree of the parameter q and the order of the derivatives, then the ellipticity condition, in the terminology of Agranovich—Vishik, becomes the condition of proper ellipticity (see (1)). As was shown in (1), if system (11) is properly elliptic in some sector Q of the plane of the complex variable q, then there exists a $q_0 > 0$ such that for all $q \in Q$ satisfying $|q| > q_0$, system (11) is uniquely solvable. Having found its solution and substituted it into (5), we obtain the solution of the original boundary-value problem.

5. Let us write the condition of proper ellipticity of system (11) in terms of the original boundary-value problem. Consider again the half-space bounded by the tangential hyperplane to Γ at the points $x_0 \in \Gamma$. Let $x' = (x_1, \dots, \dots, x_{n-1})$ and $\xi' = (\xi_1, \dots, \xi_{n-1})$ be coordinates in the hyperplane $x_n = 0$. We again write system (1) in a local coordinate system at the point x_0. Consider the boundary-value problem for the system of ordinary differential equations

$$A^{(0)}\left(x_0, \xi', -i\frac{d}{dx_n}\right) v = 0,$$

$$B^{(0)}\left(x_0, \xi', -i\frac{d}{dx_n}\right) v\Big|_{x_n=+0} = h. \tag{12}$$

* An analogous theorem holds for derivatives on the boundary in any direction.

We shall assume that ξ' is real, $q \in Q$, where Q is some sector. If, for any $x_0 \in \Gamma$ and for all (ξ', q) satisfying the condition $|\xi'| + |q| \neq 0$, system (12) is uniquely solvable in the class of functions tending to zero at infinity for $\xi' \neq 0$ or growing at infinity no faster than x_m^{m-1} for $\xi' = 0$, $q \neq 0$, then the original boundary-value problem will be called **quasi-half-bounded**. Since, by Theorem 1, the principal terms in system (11) will be the "half-space" operators obtained from $E^{(0)}_{jkpr}$, the quasi-half-boundedness of the original problem is equivalent to the half-boundedness of system (11). Hence it follows:

Theorem 2. *If assumptions I, II, III (or III′) are fulfilled and the original boundary-value problem is quasi-half-bounded in some sector Q of the plane of the complex variable q, then there exists a $q_0 > 0$ such that, for all $q \in Q$ satisfying the condition $|q| > q_0$, system (11) (and with it the original boundary-value problem) is uniquely solvable.*

6. If in assumption II the requirement of unique solvability of the Dirichlet problem "as a whole" is removed, then Theorem 2 ceases to be true. However, it is not difficult to generalize it to the case when uniqueness fails but the normal solvability of the Dirichlet problem is preserved and its index is equal to zero. In this case certain additional conditions must be imposed on the matrix g, and in formula (5) a linear combination of independent eigenfunctions of the Dirichlet problem must be added. The

arbitrary constants will be determined from the condition that the solution of a system of type (11) must satisfy the solvability condition for the Dirichlet problem. For the constants we obtain a certain system of linear algebraic equations, and the nonvanishing of the determinant of this system will be the additional condition that is to be added to Theorem 2.

The author expresses his gratitude to L. R. Volevich for a number of valuable suggestions.

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Received
4 I 1966

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Note: Figure translations are in progress. See original paper for figures.