ON THE ZERO-MANIFOLDS OF SOLUTIONS OF MULTIDIMENSIONAL DIFFERENTIAL EQUATIONS OF SECOND ORDER

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Abstract

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MATHEMATICS

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ON THE ZERO-MANIFOLDS OF SOLUTIONS OF MULTIDIMENSIONAL DIFFERENTIAL EQUATIONS OF SECOND ORDER

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Consider, in a domain G of a Euclidean space E of dimension m, the completely solvable multidimensional differential equation of second order

$$y''(x)hk + P(x)y'(x)hk + Q(x)hky(x) = 0 \qquad (h, k \in E) \tag{1}$$

with real continuous coefficients. If $y(x)$ is a solution of this equation, then $y'(x)$ and $y''(x)$ are vector and operator functions. Works ([1,2]) are devoted to the study of multidimensional linear differential equations of second order.

Throughout the entire article it is assumed that the initial data uniquely determine the solution in the whole domain G.

1. Zero-manifolds. Let $y(x)$ be some solution of equation (1). Consider the set $\mathfrak{A} = \{x : x \in G,\ y(x) = 0\}$ of all zeros of the solution $y(x)$, which we shall call the **zero-manifold**. The zero-manifold, in the topology induced by the space E, is a topological Hausdorff space. Each point of the zero-manifold $\mathfrak{A}$ has a neighborhood homeomorphic to a sphere of the Euclidean space of dimension $m-1$. It is not difficult to show that the zero-manifold $\mathfrak{A}$ is a twice differentiable manifold of dimension $m-1$. Each connected component of this manifold (and there are at most countably many of them) will be called a **zero-manifold** of the solution $y(x)$. Any compact submanifold of the domain G intersects only a finite number of zero-manifolds. The facts listed above are proved with the aid of the implicit function theorem (cf. ([3]), § 2, problem 7). For differentiable manifolds see ([4]).

2. Associated mapping. Let $y_1(x), \dots, y_m(x), y(x)$ be a fundamental system of solutions of the homogeneous equation (1). Consider the mapping $\bar{y} = \bar{y}(x) = (y_1(x), \dots, y_m(x), y(x))$ of the domain G into the coordinate Euclidean space $\bar{E}$ of dimension $m+1$. In what follows it is convenient for us to pass from this mapping to the mapping $w(x) = \bar{y}(x)/\|\bar{y}(x)\|$ of the domain G of the space E of dimension m into the unit sphere S of dimension m of the space $\bar{E}$. We shall call this mapping **associated** (with equation (1)). Let $S^0 = \{\bar{y} : \bar{y} \in S,\ y^{m+1} = 0\}$ be the "equatorial" sphere. It is clear that the zero-manifold of the solution $y(x)$ is the complete preimage of this sphere under the associated mapping. This observation makes it possible, in studying zero-manifolds, to use properties of the associated mapping.

Theorem 1. *The associated mapping $w(x)$ is a twice differentiable regular mapping of the domain G into the sphere S (regarded as twice differentiable manifolds). It realizes a local diffeomorphism. For any compact submanifold $K \subset G$ one can specify a constant $\varepsilon(K) > 0$ such that*

$$\|w'(x)h\| \geq \varepsilon(K)\|h\| \qquad (x \in G,\ h \in E), \tag{2}$$

The properties listed above are also preserved for the associated mapping considered only on the null-manifold as a mapping of this manifold into the "equatorial" sphere.

3. Properties of null-manifolds. The structure of a compact null-manifold is clarified by the following theorem.

Theorem 2. Every compact null-manifold $\mathfrak{M}$ is diffeomorphic to an $(m-1)$-dimensional sphere. In the case $m > 2$, the associated mapping w^0 (the restriction of w to $\mathfrak{M}$) realizes a diffeomorphic mapping of the null-manifold $\mathfrak{M}$ onto the sphere S^0. In the case $m = 2$, the associated mapping w^0 is a covering mapping of the null-manifold $\mathfrak{M}$ onto the circle S^0, and the number of preimages of any point of S^0 is equal to the degree of the mapping w^0, taken in absolute value.

According to the Jordan–Brouwer theorem ([5], pp. 562–564), the compact $(m-1)$-dimensional null-manifold $\mathfrak{M}$ divides the space E of dimension m into two domains $\mathfrak{M}^i$ and $\mathfrak{M}^e$ without common points, each of which has $\mathfrak{M}$ as its boundary. The domain $\mathfrak{M}^i$ is bounded and is called the interior of the manifold; the domain $\mathfrak{M}^e$ is unbounded and is called the exterior of the manifold. The degree of the mapping w^0 of the compact null-manifold $\mathfrak{M}$ onto the sphere S^0 can be computed by the formula

$$\sigma(\mathfrak{M}) = \frac{1}{\mu S^0} \int \frac{\det W\{y_1, \dots, y_m, y\}(x)}{\left(\sum_{j=1}^m y_j^2(x)\right)^{m/2} \left(\sum_{j=1}^m \left(\frac{\partial y(x)}{\partial x^j}\right)^2\right)^{1/2}}\, dx, \tag{3}$$

where the integral is taken over the manifold $\mathfrak{M}$; μS^0 is the $(m-1)$-dimensional measure of the sphere S^0; $W\{y_1, \dots, y_m, y\}(x)$ is the Wronski matrix of the system of functions $y_1(x), \dots, y_m(x), y(x)$, and the solution $y(x)$ is chosen so that the vector $y'(x)$ for $x \in \mathfrak{M}$ is directed toward the interior side of the manifold $\mathfrak{M}$ (see, for example, A. D. Myshkis' s appendix to the book of I. G. Petrovsky ([3])). In the case $m > 2$, the degree of the mapping w^0 is equal to ± 1.

Theorem 3. If the interior of the compact null-manifold $\mathfrak{M}$ of the solution $y(x)$ belongs to the domain G and $y_1(x), \dots, y_m(x), y(x)$ is a fundamental system of solutions, then the manifold $\mathfrak{M} \cap \mathfrak{M}_1 \cap \dots \cap \mathfrak{M}_j$ is diffeomorphic to an $(m-j-1)$-dimensional sphere, and the manifold $\mathfrak{M}^i \cap \mathfrak{M}_1 \cap \dots \cap \mathfrak{M}_j$ is diffeomorphic to an $(m-j)$-dimensional ball. Here $\mathfrak{M}_j$ is the null-manifold of the solution $y_j(x)$, having common points with $\mathfrak{M}$ $(j = 1, \dots, m)$.

In particular, inside the manifold $\mathfrak{M}$ there exists a unique point at which all the solutions $y_1(x), \dots, y_m(x)$ vanish.

This theorem is an analogue of the well-known Sturm theorem on the separation of zeros of solutions of a second-order differential equation.

In the course of proving Theorem 3 it is shown that

$$w(\mathfrak{M}) = S^0, \qquad w(\mathfrak{M}^i) = S^+, \tag{4}$$

and the mapping of $\mathfrak{M}^i$ onto S^+ is a diffeomorphism ($S^+ = \{\bar{y} : \bar{y} \in S,\ y^{m+1} > 0\}$). Therefore, if the interior of the compact null-manifold $\mathfrak{M}$ belongs to the domain G, then it contains no other null-manifolds of the solution $y(x)$. The arrangement of compact null-manifolds of the solution $y(x)$ obeys the following rule.

Theorem 4. Two compact null-manifolds $\mathfrak{M}$ and $\mathfrak{L}$ of one and the same solution cannot be homotopic in the domain G.

4. Theorem on the inverse mapping. In the proof of the theorems formulated above, the principal role is played by propositions of various types on the nonlocal existence of an inverse mapping. We give one of them:

Theorem 5. Let $\mathfrak{M}$ and $\mathfrak{N}$ be connected differentiable Riemannian complete manifolds, and let $w(x)$ be a differentiable mapping of the manifold $\mathfrak{M}$ into the manifold $\mathfrak{N}$.

Let, for each $x \in \mathfrak{M}$, the linear operator $w'(x)$ map the linear space $\mathfrak{M}_x$ onto the linear space $\mathfrak{N}_{w(x)}$ one-to-one and $\|w'(x)h\|_{w(x)} \geqslant \varepsilon \|h\|_x$ $(x \in \mathfrak{M},\ h \in \mathfrak{M}_x)$, where ε is some positive constant.

Then, if $\mathfrak{N}$ is simply connected, w maps $\mathfrak{M}$ onto $\mathfrak{N}$ one-to-one, and the mapping w is a diffeomorphism.

Here $\mathfrak{M}_x$ $(\mathfrak{N}_y)$ is the tangent subspace to the manifold $\mathfrak{M}$ $(\mathfrak{N})$ at the point x (y).

The proof of this theorem uses the results of the paper ([6]) and arguments usually applied in the theory of covering spaces ([7]).

5. **On nonoscillation of solutions.** We shall call a solution of equation (1) nonoscillating in a domain G if it preserves its sign in this domain.

Consider equation (1) in the ball $\|x-\xi\| \leqslant \rho$, and let

$$p = \max_{\|x-\xi\|\leqslant\rho} \|P(x)\|, \qquad q = \max_{\|x-\xi\|\leqslant\rho} \|Q(x)\|. \tag{5}$$

Theorem 6. Suppose that the condition

$$\rho < \begin{cases} \dfrac{1}{\omega} \operatorname{arc\,th} \dfrac{2\omega}{p}, & \text{if } \left(\dfrac{p}{2}\right)^2 - q = \omega^2 > 0, \\ \dfrac{2}{p}, & \text{if } \left(\dfrac{p}{2}\right)^2 - q = 0, \\ \dfrac{1}{\omega} \operatorname{arc\,tg} \dfrac{2\omega}{p}, & \text{if } \left(\dfrac{p}{2}\right)^2 - q = -\omega^2 < 0. \end{cases} \tag{6}$$

is satisfied. Then, in the ball $\|x-\xi\| \leqslant \rho$, equation (1) has a nonoscillating solution.

The proof of this theorem is based on reducing equation (1) to the multidimensional Riccati equation

$$z'(x) = -Q(x) - P(x)z(x) - z(x) \otimes z(x) \tag{7}$$

(by the substitution $z(x) = y'(x)/y(x)$) and on using the theorem on integral inequalities ([8]).

In conclusion we give one assertion of a metric character, concerning zero sets.

Theorem 7. Let the points $\xi_1, \dots, \xi_{m+1}$ belong to the zero set of a nontrivial solution of equation (1), and suppose that the system of points $\xi_1, \dots, \xi_{m+1}$ is independent.

Then the inequality

$$(r^2 - \delta^2)q + \max_{1\leqslant i\leqslant m+1} \left(\sum_{j=1}^{m+1} \frac{\|\xi_i - \xi_j\|^2}{r_j} \right) p > 2, \tag{8}$$

holds, where r is the radius of the ball circumscribed about the simplex $\sigma = (\xi_1, \dots, \xi_{m+1})$, δ is the distance from σ to the center of the circumscribed ball, and r_j is the distance from the vertex ξ_j to the $(m-1)$-dimensional plane passing through the opposite face of the simplex.

The proof is based on one proposition from ([9]).

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Note: Figure translations are in progress. See original paper for figures.