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MATHEMATICS

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Abstract

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MATHEMATICS

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ON A PROBLEM OF MIXED TYPE FOR THE EQUATION $y(y-1)u_{xx} + u_{yy} = 0$

(Presented by Academician M. A. Lavrent' ev on 11 V 1965)

The equation

$$L(u) \equiv k(y)u_{xx} + u_{yy} = 0, \tag{L}$$

where $k(y) = y^2 - y$, is an equation of mixed type; it is elliptic for $y < 0$ and $y > 1$, hyperbolic for $0 < y < 1$, and along the straight lines $y = 0,\ y = 1$ it degenerates parabolically. The families of curves $x \pm {}^1\!/_4 k' \sqrt{-k} \pm {}^1\!/_8 \arcsin k' = \text{const}$ are real characteristics of equation (L), while the straight lines $y = 0,\ y = 1$ constitute the locus of points of return of this family.

1. We introduce the following notation:

$$\omega(y) = {}^1\!/_4 k' \sqrt{-k} + {}^1\!/_8 \arcsin k', \qquad \tilde{y}(y) = -(3/2a)^{2/3} (\omega \mp \pi/16)^{2/3},$$

$$0 \le y \le 1, \tag{1}$$

$$\omega(y) = {}^1\!/_4 k' \sqrt{k} - {}^1\!/_8 \ln |k' + 2\sqrt{k}|, \qquad \tilde{y}(y) = (3\omega/2a)^{2/3},$$

$$y \le 0, \quad y \ge 1;$$

$$\xi(x, y) = (x + \omega - \pi/16)/a, \qquad \eta(x, y) = (x - \omega + \pi/16)/a,$$

$$a = \text{const}, \quad \tilde{x} = x/a; \tag{2}$$

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$$\pi/8 < a < \pi/4, \qquad b(\tilde{y}) = [\tilde{y}'(y)]^{-2}\tilde{y}'''(y), \qquad \tilde{\omega}(y) = \exp\left({}^{1}\!/_{2}\int_0^{\tilde{y}} b(s)\,ds\right).$$

In the plane of the variables x, y, consider the points $A_1(0,0)$, $A_0(\pi/8, 0)$, $A_2(a, 0)$, $B(a - \pi/16, {}^1\!/_2)$, $A_3(a, 1)$, $A(a - \pi/8, 1)$, $A_4(0, 1)$, $B_1(\pi/16, {}^1\!/_2)$. Let D^* be a bounded simply connected domain bounded by: 1) the arc σ_0: $(x - a/2)^2 + \omega^2 = a^2/4,\ y \le 0$; 2) the arc σ_1: $(x - a/2)^2 + \omega^2 = a^2/4,\ y \ge 1$; 3) the characteristics $A_1B_1 : \eta = \pi/8a$, $B_1A_4 : \xi = 0$, $A_2B : \xi = 1 - \pi/8a$, $BA_3 : \eta = 1$ of equation (L). Let, further, $B_0(D)$ be the point of intersection of the characteristics $A_0B_0 : \eta = \pi/4a$ and $AB_0 : \xi = 1 - \pi/8a$ ($AD : \eta = 1 - \pi/8a$ and B_1A_4); D be the subdomain of D^* lying above the curve DAB; D^+ the elliptic part of the domain D; Δ (Δ^*) the hyperbolic part of D lying above the curve A_4DA (ABA_3); D^- the domain of the characteristic quadrilateral AB_0A_0D.

Following A. V. Bitsadze ([1]), by a solution of equation (L) that is regular in the domain D^* we shall mean a solution $u(x, y)$ possessing the following properties: 1) u is continuous everywhere in the closed domain $\overline{D^*}$, except at the points A, A_0, A_2 and, possibly, at the points A_1, A_3, A_4, where it tends to ∞ of order $\alpha < {}^1\!/_6$; 2) u_y (u_x) is continuous everywhere in $\overline{D^*}$, except, possibly, at the points $A, A_i,\ i = 0, \dots, 4$, and at the characteristics issuing from these points, where it tends to ∞ of order $< {}^5\!/_6$ ($< {}^7\!/_6$) and < 1 (< 1), respectively; 3) u is twice continuously differentiable everywhere in D^*, except, possibly, at the boundary of the domain D^-.

Problem 1. Find a regular solution in the domain D^* of equation (L), satisfying the conditions:

$$(\tilde{\omega}u)_{\sigma_0} = \varphi_0(\tilde{x}), \quad (\tilde{\omega}u)_{\sigma_1} = \varphi_1(\tilde{x}), \quad 0 < \tilde{x} < 1; \tag{3}$$

$$(\tilde{\omega}u)_{AB_0} = \psi_1(\eta), \quad x_0 < \eta \le \pi/4a; \quad (\tilde{\omega}u)_{AD} = \psi_2(\xi),$$

$$0 \le \xi < x_0 = 1 - \pi/8a,$$

where $\tilde{\omega}$ is given by formulas (1), (2), with in (1) the minus sign taken before $\pi/16$, and φ_i and ψ_i are representable in the form

$$\varphi_0(\tilde{x}) = \alpha_0(\tilde{x})\tilde{x}^{-\chi_0}(1 - \tilde{x})^{-\alpha/2}, \quad \varphi_1(\tilde{x}) = \alpha_1(\tilde{x})(\tilde{x} - \tilde{x}^2)^{-\chi_1},$$

$$\psi_1(\eta) = \beta_1(\eta)(\eta - x_0)^{-\alpha}(\pi/4a - \eta)^{1/6-\alpha}, \quad \psi_2(\xi) = \beta_2(\xi)(x_0 - \xi)^{-\alpha},$$

where

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$$\alpha_i \in C^{(2)} \quad (0 \le \tilde{x} \le 1), \quad \alpha_0(1) \ne 0; \quad \beta_1 \in C^{(4)} \quad (x_0 \le \eta \le \pi/4a),$$

$$\beta_i(x_0) \ne 0, \quad \beta_1(\pi/4a) \ne 0, \quad \beta_2 \in C^{(4)} \quad (0 \le \xi \le x_0), \quad 1/12 > \chi_i = \text{const}.$$

2. Uniqueness of the solution of Problem 1 follows directly from the uniqueness of the solution of the Tricomi problem for equation (L) in the domain D^- with data on the characteristics AB_0, AD, and from the following two lemmas.

Lemma 1. Let $u(x, y)$ be a regular solution in the domain Δ (Δ^*) of equation (L), vanishing on AD (AB). Then

$$\int_0^{x_1} u(x,1)u_y(x,1)\,dx \ge 0 \quad \left(\int_{x_1}^{a} u(x,1)u_y(x,1)\,dx \ge 0\right), \quad x_1 = a - \pi/8.$$

Lemma 2. Let $u(x, y)$ be a regular solution in the domain D^+ of equation (L), vanishing on σ_1. Then

$$\int_0^{a} u(x,1)u_y(x,1)\,dx \le 0.$$

The validity of Lemma 1 is established by the method of F. I. Frankl ([2]). Lemma 2 is proved in the same way as in the case of the Tricomi equation ([2]).

3. Proof of existence of the solution.
Lemma 3. Let: 1) $u \in C(\overline{D^+})$; 2) $u \in C^{(1)}(D^+ - A_3 - A_4)$; 3) $u \in C^{(2)}(D^+)$, $L(u) \equiv 0$ in D^+; 4) $(u)_{\sigma_1} = 0$, $u_y(x,1) + {}^1/_5 u(x,1) = 0$ for $0 < x < a$. Then $u \equiv 0$ in $\overline{D^+}$.

Lemma 4. Let u satisfy the conditions of Lemma 1 and

$$u_y(x,1) + {}^1/_5 u(x,1) = 0$$

for $0 < x < x_1$. Then $u \equiv 0$ in Δ.

Lemma 3 follows simply from the known Zaremba–Giraud extremum principle ([3,4]), and Lemma 4 is a consequence of Lemma 1.

In equation (L) we pass to Frankl' s variables ([5]) $\tilde{x}, \tilde{y}$ and to the new function $v = \tilde{\omega} u$. Then v will satisfy the equation

$$\tilde{y} v_{\tilde{x}\tilde{x}} + v_{\tilde{y}\tilde{y}} + c(\tilde{y}) v = 0, \qquad (\widetilde{L})$$

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where

$$c(\tilde{y}) = -{}^1/_2 \left(b_{\tilde{y}} + {}^1/_2 b^2\right),$$

and the domain D (D^+, Δ, Δ^*) passes into the domain $\widetilde{D}$ $(\widetilde{D}^+, \widetilde{\Delta}, \widetilde{\Delta}^*)$, bounded by the normal contour

$$\tilde{\sigma}_1 : \quad (\tilde{x} - {}^1/_2)^2 + {}^4/_9 \tilde{y}^3 = {}^1/_4$$

and the characteristics $\widetilde{A}_4\widetilde{D}$, $\widetilde{D}\widetilde{A}$, $\widetilde{A}\widetilde{B}$, $\widetilde{B}\widetilde{A}_3$ of equation $(\widetilde{L})$.

It is easy to show that

$$\tilde{y} = (y-1)a^{-2/3}F^{2/3}\left(-{}^1/_2, {}^3/_2, {}^5/_2, 1-y\right)$$

in $\overline{D}$; consequently, $c(\tilde{y})$ for

$$\tilde{y} > -(3\pi/16a)^{2/3} \quad (y > 0)$$

by virtue of (2) admits derivatives of any order, and $c(0) > 0$.

Problem (1,1). Find a regular solution in the domain $\widetilde{D}^+$ of equation $(\widetilde{L})$, satisfying the conditions:

$$(v)_{\tilde{\sigma}1} = \varphi_1(\tilde{x}) \quad \text{for} \quad 0 < \tilde{x} < 1,$$

$$(v_{\tilde{y}})_{\tilde{y}=0} = \nu(\tilde{x}) \quad \text{for} \quad 0 < \tilde{x} < x_0, \quad x_0 < \tilde{x} < 1.$$

Problem (1,2). Find a regular solution in the domain $\widetilde{\Delta}$ $(\widetilde{\Delta}^*)$ of equation $(\widetilde{L})$, satisfying the conditions

$$(v)_{\widetilde{AD}} = \psi_2(\xi) \quad \text{for} \quad 0 \leq \xi < x_0;$$

$(v\tilde{y})_{\tilde{y}=0} = \nu(\tilde{x})$ for $0 < \tilde{x} < x_0$ ($v = \psi_1(\eta)$ on $\widetilde{AB}$ for $x_0 < \eta \leqslant 1$; $v_{\tilde{y}}^2 = \nu(\tilde{x})$ for $\tilde{y} = 0$ and $x_0 < \tilde{x} < 1$).

In both problems it is assumed that $\nu(\tilde{x}) \in C^{(1)}$ $(0 < \tilde{x} < x_0,\ x_0 < \tilde{x} < 1)$, but may tend to infinity of order $< 5/6$ for $\tilde{x} = 0, x_0, 1$.

The uniqueness of the solution of problems (1,1) and (1,2) is established with the aid of Lemmas 3 and 4, while the proof of existence is carried out according to the scheme proposed by Gellerstedt in [6].

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Lemma 5. *Suppose: 1) there exist solutions of Problem 1 in the domain D; 2)* $\tau(x) = (v)_{\tilde{y}=0}$, $\nu(x) = (v_{\tilde{y}})_{\tilde{y}=0}$. *Then*

$$\tau(x) = \gamma \int_0^1 \nu(t) \left[L(x,t) - |t-x|^{-1/3} + (t+x-2tx)^{-1/3}\right] dt + \Phi(x), \qquad 0 < x < 1; \tag{4}$$

$$\tau(x) = \gamma \int_x^{x_0} \nu(t)(t-x)^{-1/3} P((t-x)^{4/3})\, dt + \Psi_2(x), \qquad 0 < x < x_0; \tag{5}$$

$$\tau(x) = \gamma \int_{x_0}^{x} \nu(t)(x-t)^{-1/3} P((x-t)^{4/3})\, dt + \Psi_1(x), \qquad x_0 < x < 1; \tag{6}$$

where: 1) $\gamma \equiv \text{const}$, $L \in C$ $(0 \le x, t \le 1)$, $L \in C^{(2)}$ $(0 < x, t < 1)$; *moreover for* $x = t$ $(x = t = 0, 1)$ *the first derivatives may tend to infinity as* $\ln|x-t|$ $((t+x-2tx)^{-1/3})$; *2)* $\Phi \in C^{(3)}$ $(0 < x < 1)$ *and for* $x = 0, 1$ *tends to infinity of order* $2\chi_1$; *3)* $P((x-t)^{4/3})$ *admits derivatives of any order for* $x \neq t$, *moreover* $P(0) = 1$, *and the first derivatives for* $x = t$ *tend to zero of order not less than* $1/3$; *4)* $\Psi_2 \in C^{(3)}$ $(0 \le x < x_0)$, $\Psi_1 \in C^{(3)}$ $(x_0 < x \le 1)$, *and for* $x = x_0$ Ψ_i *tends to infinity of order* α.

Lemma 5 follows directly from the constructive properties of the solution of problems (1,1) and (1,2).

From (4), (5), and (6), after a series of transformations entirely analogous to the transformations used in the case of the Tricomi problem, for determining $\nu(x)$ we obtain the singular integral equation

$$\nu(x) = \lambda \int_0^{x_0} \nu(t) \left[\left(\frac{t-x_0}{x-x_0}\right)^{2/3} \frac{1}{t-x} + \left(\frac{t+x_0-2tx_0}{x-x_0}\right)^{2/3} \frac{1}{t+x-2tx}\right] dt -$$

$$-\lambda \int_{x_0}^{1} \nu(t) \left[\left(\frac{t-x_0}{x-x_0}\right)^{2/3} \frac{1}{t-x} - \left(\frac{t+x_0-2tx_0}{x-x_0}\right)^{2/3} \frac{1}{t+x-2tx}\right] dt + \tag{7}$$

$$+\lambda \int_0^1 \nu(t) k(x,t)\, dt + h(x), \qquad \lambda = \frac{1}{\pi\sqrt{3}},$$

where $h(x) \in C^{(2)}$ $(0 < x < x_0,\ x_0 < x < 1)$; $h(x)$ for $x = x_0$ tends to infinity of order $\alpha + 2/3$ and to infinity of the same order at the points $x = 0,\ x = 1$,

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if $2\chi_1 = \alpha$; the kernel $k(x,t)$ has the following properties: 1) it is regular for $0 < x, t < 1$ and $x \neq t$; for $x = t$ the first derivatives may tend to infinity of order $\leq \varepsilon + 2/3$; 2) for $0 < x, t < 1,\ x \neq x_0$, it satisfies the inequality

$$|k(x,t)| \leq M_1 (x - x_0)^{-2/3} [|L(x_0, t)| +$$

$$+ |x - x_0|^{1-\varepsilon} |\ln(t + x - 2tx)|] + M_2,$$

where $0 < M_i = \text{const}$, and ε is an arbitrarily small positive number.

By the change of variable of integration $t = (s+1)/2$, equation (7) can be rewritten in the form

$$\tilde{v}(y) = \lambda \int_{-1}^{y_0} \tilde{v}(s) \left[\left(\frac{s - y_0}{y - y_0} \right)^{2/3} \frac{1}{s - y} + \left(\frac{1 - s y_0}{y - y_0} \right)^{2/3} \frac{1}{1 - s y} \right] ds - \quad (8)$$

$$-\lambda \int_{y_0}^{1} \tilde{v}(s) \left[\left(\frac{s - y_0}{y - y_0} \right)^{2/3} \frac{1}{s - y} - \left(\frac{1 - s y_0}{y - y_0} \right)^{2/3} \frac{1}{1 - s y} \right] ds + \lambda \int_{-1}^{1} \tilde{v}(s) \tilde{k}(y, s)\, ds + \tilde{h}(y),$$

where $y = 2x - 1,\ y_0 = 2x_0 - 1,\ \tilde{v}(y) = v((y+1)/2)$.

Equation (8) for $\tilde{k}(y,s) \equiv 0$ was studied by Gellerstedt (7). Using his results and relying on the uniqueness of the solution of problems 1; 1.1; 1.2, it is not difficult to show that equation (7) is unconditionally solvable (in the class of sought solutions) and that its solution $v(x) \in C^{(1)}$ $(0 < x < x_0,\ x_0 < x < 1)$, moreover it becomes infinite of order $\alpha + 2/3$ for $x = 0,\ x_0,\ 1$.

Knowing $v(x)$ from (7), with the help of problems 1.1 and 1.2 we prove the existence of a solution of problem 1 in the domain D.

The construction of the solution $u(x,y)$ of problem 1 in the domain D^- presents no essential difficulty, and therefore we restrict ourselves to giving the final result.

There exists a solution $u(x,y)$ of problem 1 in the domain D^-, and it has the following properties: 1) $u \in C(\bar{D}^- - A - A_0)$, and at the points $A,\ A_0$ it becomes infinite of order α; 2) $\partial u/\partial \xi$ is continuous everywhere in the closed domain $\bar{D}^-$, except for the characteristic AB_0 and the point A_0, where it becomes infinite of order $\alpha + 5/6$ for $y \neq 1$ and infinite of order $\alpha + 1$ at the points $A,\ A_0$; 3) $\partial u/\partial \eta$ is continuous everywhere in $\bar{D}^-$, except for the characteristics $AD,\ A_0B_0$, where it becomes infinite of order $\alpha + 5/6$ for $y \neq 1,\ y \neq 0$ and infinite of order $\alpha + 1$ at the points $A,\ A_0$; 4) let $\widetilde{(\omega u)}_{A_0B_1} = \tilde{\psi}_1(\eta),\ \pi/8a \leq \eta < \pi/4a$; $\widetilde{(\omega u)}_{A_0B_0} = \tilde{\psi}_2(\xi),\ 0 < \xi \leq x_0$, while ω is given by formulas (1), (2), where before $\pi/16$ the sign + remains. Then

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$$\tilde{\psi}_1^{(i)}(\eta) = b_i(\eta)(\pi/4a - \eta)^{-\alpha-i}, \qquad i = 0, 1, 2, 3, 4;$$

$$\tilde{\psi}_2^{(i)}(\xi) = a_i(\xi)\xi^{-\alpha-i}(x_0 - \xi)^{1/6-\alpha-i}, \qquad i = 1, 2, 3, 4;$$

$$\tilde{\psi}_2^{(i)}(\xi) = a_0(\xi)\xi^{-\alpha}, \qquad i = 0,$$

where

$$b_i \in C\left(\pi/8a \le \eta \le \pi/4a\right), \qquad a_i \in C(0 \le \xi \le x_0), \qquad b_i(\pi/4a) \ne 0,$$

$$a_i(0) \ne 0,$$

(i) denotes the order of the derivative.

The construction of the solution $u(x, y)$ of problem 1 in the domain $D^* - (D + D^-)$ is carried out in the same way as in the domain D, proceeding only from the enumerated properties of the function u on the characteristics A_0B_1 and A_0B_0.

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