ON EXISTENCE THEOREMS IN THE CALCULUS OF VARIATIONS

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Abstract

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ON EXISTENCE THEOREMS IN THE CALCULUS OF VARIATIONS

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In this note an existence theorem is formulated for time-optimal processes, analogous to the existence theorem of Filippov–Warga–Roxin (see ([1–3])), but in a somewhat more general form than the original formulation of this theorem. It is then shown that the classical Hilbert–Tonelli existence theorem for the basic problem of the calculus of variations, as well as the existence theorem for the Lagrange problem (see ([4])), follows from the proposition formulated here.

1. Formulation of the existence theorem. Let $f = (f^1, \dots, f^n) = f(x, u)$ be an n-dimensional vector function of the variables x, u, continuously differentiable with respect to x and continuous with respect to u; here $x = (x^1, \dots, x^n)$ ranges in a given open set G of the n-dimensional space R^n, and u ranges in a given **compact** set U of an r-dimensional space. We shall call a **control** an arbitrary measurable function of the real argument t with values in U; t will be regarded as time.

Let us write the differential equation

$$dx/dt = f(x, u), \tag{1}$$

regarding u in it as a parameter. We shall say that to a control $u(t)$, $t_1 \le t \le t_2$, there corresponds a trajectory $x(t)$, $t_1 \le t \le t_2$, of equation (1), or that the control $u(t)$ carries the phase point x along a trajectory of equation (1) from $x(t_1)$ to $x(t_2)$, if for almost all t, $t_1 \le t \le t_2$, $dx/dt = f(x(t), u(t))$.

Let two **compact sets** A_1, A_2, lying in G, be given. The time-optimal problem with boundary conditions determined by the sets A_1, A_2 consists in choosing a control $u(t)$, $t_1 \le t \le t_2$, carrying the phase point along a trajectory $x(t)$, $t_1 \le t \le t_2$, of equation (1) from A_1 to A_2 ($x(t_1) \in A_1,\ x(t_2) \in A_2$) in the least time $t_2 - t_1$.

We shall say that equation (1) satisfies the **compactness condition** if, for any sequence of controls $u_i(t)$, $t_1^{(i)} \le t \le t_2^{(i)}$, $i = 1, 2, \dots$, carrying the phase point along trajectories $x_i(t)$ from A_1 to A_2 $(x(t_1) \in A_1,\ x(t_2) \in A_2)$, there exists a compact set $K \subset G$ such that all the trajectories $x_i(t)$ are contained in K.

Introduce the notation $\psi = (\psi_1, \dots, \psi_n)$,

$$H(\psi, x, u) = \sum_{\alpha=1}^{n} \psi_\alpha f^\alpha(x, u) = \psi \cdot f(x, u)$$

and consider the following equation with respect to u (for fixed ψ, x):

$$H(\psi, x, u) = M(\psi, x), \tag{2}$$

where

$$M(\psi, x) = \max_{u \in U} H(\psi, x, u).$$

Let us denote by $\Omega_{\psi,x}$ the set of all nonnegative solutions of equation (2). Finally, let $F_{\psi,x}$ denote the image of the set $\Omega_{\psi,x}$ in R^n under the mapping $f(x, u)$: $\Omega_{\psi,x} \to R^n$.

Existence theorem. Suppose that equation (1) satisfies the compactness condition and that the sets $F_{\psi,x}$ are convex. Then, if there exists at least one control that carries the phase point along a trajectory of equation (1) from A_1 to A_2, there also exists an optimal control carrying the phase point from A_1 to A_2 in minimum time.

In particular, $F_{\psi,x}$ is convex if $\Omega_{\psi,x}$ consists of a single point or is empty, i.e. if equation (2) has at most one nonnegative solution.

Proof of the theorem. Consider the controlled equation

$$\frac{dx}{dt} = \sum_{\alpha=0}^{n} p_\alpha f(x, u_\alpha) = g(x, p, v), \tag{3}$$

in which the control vector is the vector (p, v), where $p = (p_0, \dots, p_n)$ belongs to the n-dimensional simplex

$$T^n = \left\{ p = (p_0, \dots, p_n),\ \sum_{\alpha=0}^{n} p_\alpha = 1,\ p_\alpha \geq 0 \right\},$$

and the point $v = (u_0, \dots, u_n) \in U^{n+1}$, the $(n+1)$-st topological power of the set U.

Consequently, a control for equation (3) is any measurable function with values in the compact set $W = T^n \times U^{n+1}$. For fixed x, the image of W in R^n under the mapping $g(x,p,v)$ is convex. Indeed, if N_x denotes the image of U in R^n under the mapping $f(x,u)$, then the image of W in R^n under the mapping $g(x,p,v)$ coincides with the union of the convex hulls of arbitrary $n+1$ points taken from N_x. Therefore, for equation (3) the existence theorem is valid (see ([1–3])). Every optimal control and the corresponding trajectory of equation (3) satisfy the maximum principle (see ([5])), i.e. the system (4)–(5):

$$\frac{dx_i}{dt} = g(x,p,v), \qquad \frac{d\psi_i}{dt} = -\frac{\partial}{\partial x_i}\psi g(x,p,v), \tag{4}$$

and for almost all t

$$\psi(t)g(x(t),p(t),v(t)) = \sum_{\alpha=0}^{n} p_\alpha(t)H(\psi(t),x(t),u_\alpha(t)) = \max_{(p,v)\in T^n\times U^{n+1}} \psi(t)g(x(t),p,v) \geq 0. \tag{5}$$

It follows from equation (5) that, for almost all t,

$$H(\psi(t),x(t),u_\alpha(t)) = M(\psi(t),x(t)) \geq 0, \qquad \alpha = 0,1,\dots,n.$$

Consequently, according to the assumption, for almost all t

$$\sum_{\alpha=0}^{n} p_\alpha(t)f(x(t),u_\alpha(t)) \in F_{\psi(t),x(t)}.$$

Therefore there exists a control $u(t)$ such that, for almost all t,

$$f(x(t),u(t)) = \sum_{\alpha=0}^{n} p_\alpha(t)f(x(t),u_\alpha(t));$$

it is the optimal control.

2. Existence theorem for the basic problem of the calculus of variations.

Let the functional

$$\int_{t_1}^{t_2} L(t,y,\dot{y})\,dt, \qquad y = (y^1,\dots,y^n),$$

be minimized, where $\dot{y} = dy/dt$, defined for values of y,t belonging to a given closed set G of $(n+1)$-dimensional space, and for arbitrary values of $\dot{y}$. It is assumed that L is continuously differentiable with respect to y,t and twice continuously differentiable with respect to $\dot{y}$. The boundary conditions are given by compact sets A_1, A_2 from G: $(t_1, y(t_1)) \in A_1$, $(t_2, y(t_2)) \in A_2$.

Existence theorem. *Suppose that, for the values of the arguments under consideration,*

$$L > 0, \quad \Delta = \left|\partial^2 L/\partial \dot{y}_i \partial \dot{y}_j\right| \neq 0, \quad \lim_{|\dot{y}|\to\infty} |\dot{y}|/L(t, y, \dot{y}) = 0. \tag{6}$$

Then in G there exists an absolutely continuous curve $y(t)$ satisfying the prescribed boundary conditions and minimizing the integral

$$\int_{t_1}^{t_2} L\,dt.$$

Proof. Introduce the time

$$\tau = \int_{t_1}^{t} L\,ds$$

and denote $x = (t, y)$. Then the problem under consideration is obviously equivalent to the following time-optimal control problem:

$$dx/dt = f(x, u), \qquad \text{where } f(x, u) = (1/L(x, u),\ u/L(x, u))$$

and the control parameter u varies in n-dimensional space compactified by adjoining the infinitely distant point to the n-dimensional topological sphere (according to (6), for $u = \infty$ we have $f(x, u) = 0$). To prove the existence of a minimizing function, it is enough to show that the set $\Omega_{\psi,x}$ consists of a single point, whatever the vectors ψ and $x \in G$. We have

$$H(\psi, x, u) = \frac{\psi_1}{L(x, u)} + \sum_{\alpha=1}^{n} \frac{\psi_{\alpha+1} u^\alpha}{L(x, u)}.$$

It is easy to exclude the case $\psi_1 = -1$, $\psi_2 = \dots = \psi_{n+1} = 0$; then $H(\psi, x, u) = M(\psi, x)$ for finite values of u. For these u we have $\partial H/\partial u^i = 0$, i.e. $\psi_{i+1} = H(x, \psi, u)L_{u_i}$, $i = 1, \dots, n$. From the condition $\Delta \neq 0$ it follows that this system (with respect to u) has only isolated solutions, and the uniqueness of the solution for the equation $H(\psi, x, u) = M(\psi, x)$ follows from this and from the simple connectedness of the n-dimensional sphere for $n \geq 2$.

The theorem can also be formulated in a more general form by replacing the condition $\Delta \neq 0$ by the condition of convexity of the function L with respect to $\dot{y}$. Finally, in an analogous way one can prove the following existence theorem for the Lagrange problem.

3. Existence theorem for the Lagrange problem.

Let k functions $f_i(t, y_1, \dots, y_n, v_1, \dots, v_{n-k})$, $i = 1, \dots, k$, be given. The vector-function $f(t, y, v)$ is continuously differentiable with respect to $(t, y) \in G$ and for arbitrary values of the vector $v = (v_1, \dots, v_r)$, $r = n - k$, where G is a closed set of $(n+1)$-dimensional space. Consider the system of equations

$$dy_i/dt = f_i(t, y_1, \dots, y_n, dy_{k+1}/dt, \dots, dy_n/dt);$$
$$dy_{k+j}/dt = v_j, \qquad i = 1, \dots, k;\ j = 1, \dots, n-k. \tag{7}$$

The function $L(t, y, \dot{y})$ is defined, as above, in the basic variational problem

$$L(t, y, v) = L(t, y, f_1(t, y, v), \dots, f_k(t, y, v), v).$$

The Lagrange problem consists in minimizing the functional $\int_{t_1}^{t_2} L(t, y, \dot{y})\, dt$, where y satisfies the given system (7); the boundary conditions, as above, are specified by compact sets A_1 and A_2.

Existence theorem. *Suppose that, for the values of the arguments under consideration,*

$$L > 0, \qquad \lim_{|\dot{y}| \to \infty} |\dot{y}|/L(t, y, \dot{y}) = 0, \qquad \Delta = |\partial^2 L/\partial v_i \partial v_j| \neq 0.$$

Suppose also that, for fixed (t, y), f_i is a linear function of v.

Then in G there exists an absolutely continuous curve $y(t)$ satisfying system (7) and the prescribed boundary conditions and minimizing the functional.

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Note: Figure translations are in progress. See original paper for figures.