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ON SOLUTIONS IN GENERALIZED FUNCTIONS OF SOME ORDINARY DIFFERENTIAL EQUATIONS WITH POLYNOMIAL COEFFICIENTS

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As is known ((1); see also the notation there), in the space K' the number of linearly independent solutions of an ordinary differential equation of order p with singularities in the coefficients may be different from p. Thus, for example, the equation

$$xy' = \lambda y \tag{1}$$

has 2 solutions in K', while the equation

$$x^3 y' = 2y \tag{2}$$

has no solution at all in this space ((1), p. 61).

In the paper (2) it is proved that an equation of the Fuchs class of order n has in K' exactly $2n$ linearly independent solutions. In our paper (3), for the Euler equation, all $2n$ solutions are written out explicitly.

In the present note equations of the form

$$Ly = \sum_{k=0}^{m} a_k x^{k+r} y^{(k)}(x) + \sum_{q=0}^{n} b_q x^q y^{(q)}(x) = 0, \quad r > 0, \quad m > n\text{—integers} \tag{3}$$

are investigated (not of the Fuchs class). Since classical solutions of such equations near a singular point may have exponential growth (for example, $y = e^{-1/x^2}$ is a classical solution of equation (2)), the space of generalized functions is taken to be the space $(S_0^\beta)'$ (with some $\beta > 1$), where regularization of functions with exponential singularities is allowed (4).

Let Ω denote the subspace of solutions of equation (3) in $(S_0^\beta)'$. Let Ω_0 denote the totality of functionals from Ω concentrated at zero. Let, further,

$$[\lambda]_\rho = \lambda(\lambda-1)\dots(\lambda-\rho+1), \qquad Q(\lambda) = \sum_{k=0}^{m}(-1)^{(k+r)}a_k[\lambda+k+r]_{k+r},$$

$$R(\lambda) = \sum_{k=0}^{n}(-1)^k b_k[\lambda+k]_k.$$

Lemma. *If the polynomial $R(\lambda)$ has no positive integer roots, then the dimension of Ω_0 in $(S_0^\beta)'$ is equal to r.*

Proof. A functional concentrated at zero in $(S_0^\beta)'$ has the general form (5)

$$y = \sum_{i=0}^{\infty} c_i \delta^{(i)}(x). \tag{4}$$

It remains to check how many arbitrary constants c_i there are in this series that represents a solution of equation (3). Substituting the series (4) into equation (3)

and using the formulas $x^k\delta^{(q)}(x) = (-1)^k[q]_k\delta^{(q-k)}(x)$ for $k < q$ $(= 0$ for $k > q)$, by virtue of the linear independence of $\delta^{(i)}(x)$ for different i, to determine the coefficients c_i we obtain:

$$Q(i)c_{i+r} + R(i)c_i = 0, \quad i = 0, 1, \dots \tag{5}$$

We split the system (5) into r subsystems: the first is separated from (5) for the values $i = 0, r, 2r, \dots, kr, \dots$; the second for $i = 1, 1+r, 1+2r, \dots, 1+kr, \dots$; the r-th for $i = r-1, 2r-1, \dots, kr-1, \dots$. We shall now prove that in each subsystem there is one and only one free c_{i_0}, and all the remaining c_i are determined uniquely through it. For definiteness, let us take the first subsystem.

If $Q(i) \neq 0$ for all $i = 0, r, 2r, \dots$, then, obviously, assigning c_0 arbitrarily, it is easy to determine the subsequent c_i; moreover, it is not hard to see that

$$|c_{(k+1)r}| \leq A^k/k^{k(m+r-n)}, \quad k \to \infty, \ A = \text{const.} \tag{6}$$

Consider the case where $Q(i) = 0$ for some $i = 0, r, 2r, \dots$, and let $k_0 r$ be the largest integral positive root of the polynomial $Q(i)$ in the first subsystem. Assigning $c_{k_0 r}$ arbitrarily, it is easy to see that all subsequent c_{kr} are expressed through it uniquely, while all preceding c_{kr} are equal to zero. In this case the c_{kr} satisfy the estimate (6). In an analogous manner we are convinced that in each subsystem there is only one free c_{i_0}, while the remaining c_i are expressed through it uniquely and satisfy the estimate (6) as $i \to \infty$.

Let us verify that the series (4) converges in some $(S_0^\beta)'$. Represent the series (4) as the sum of r series, each of which is a solution of the corresponding subsystem. For the first subsystem we have

$$\left|\sum_{k=0}^{\infty}(c_{kr}\delta^{(kr)}(x), \varphi(x))\right| \leq \sum_{k=0}^{\infty}|c_{kr}|\,|\varphi^{(kr)}(0)| \leq \sum_{k=0}^{\infty}\frac{B^k}{k^{k(m+r-n-\beta r)}} < \infty$$

for $1 < \beta < (m+r-n)/r$. The convergence of the series (4) in each subsystem is checked analogously.

Remark 1. It is possible to prove that the assertion of the lemma is also true under the assumption that all integral positive roots $R(\lambda)$ in each subsystem are larger than the largest integral positive root of $Q(\lambda)$.

Definition. We shall say that a polynomial $A(\lambda)$ **has no generalized multiple roots modulo** r if the arithmetic progressions with difference r, constructed on the roots of the polynomial $A(\lambda)$, do not intersect. If an arithmetic progression constructed over a root λ_0 with difference r contains k roots of the polynomial $A(\lambda)$, then we shall say that λ_0 is a **generalized multiple root modulo** r of $A(\lambda)$ **of order** k.

It is obvious that ordinary multiple roots are generalized multiple roots, but the converse is false.

Put

$$f^\lambda(x) = \frac{|x|^\lambda}{\Gamma((\lambda+1)/2)}, \quad g^\lambda(x) = \frac{|x|^\lambda \operatorname{sign} x}{\Gamma((\lambda+2)/2)}.$$

As is known ([1], p. 81 and following), by analytic continuation these functions define a functional for all values of λ, in particular

$$f^{-(2n+1)}(x) = \frac{(-1)^n n!\,\delta^{(2n)}(x)}{(2n)!}, \quad g^{-2n}(x) = \frac{(-1)^n (n-1)!\,\delta^{(2n-1)}(x)}{(2n-1)!}.$$

Put

$$A(\lambda) = \sum_{k=0}^{m} a_k 2^k \left[\frac{\lambda - r}{2}\right]_{E((k+1)/2)} \left[\frac{\lambda - 1}{2}\right]_{E((k+r)/2)},$$

$$B(\lambda)=\sum_{k=0}^{m} a_k 2^k \left[\frac{\lambda-r-1}{2}\right]_{E(k/2)} \left[\frac{\lambda}{2}\right]_{E((k+r+1)/2)},$$

$$C(\lambda)=\sum_{k=0}^{n} b_k 2^k \left[\frac{\lambda}{2}\right]_{E((k+1)/2)} \left[\frac{\lambda-1}{2}\right]_{E(k/2)},$$

where $E(\lambda)$ is the integer part of the real number λ.

Theorem 1. *If the roots of the polynomial $C(\lambda)$ are not comparable modulo r with the roots of $A(\lambda)$ and $B(\lambda)$, and the polynomials $A(\lambda)$ and $B(\lambda)$ have no common multiple roots modulo r, then equation (3) in $(S_0^\beta)'$ $(1<\beta<(m+r-n)/r)$ has $(2m+r)$ linearly independent solutions of the form*

$$y=\sum_{\lambda}\xi_\lambda f^\lambda(x)+\sum_{\lambda}\eta_\lambda g^\lambda(x), \tag{7}$$

where λ ranges over certain arithmetic progressions.

Proof. It suffices to show that in the expansion (7) there are $(2m+r)$ free coefficients allowing one to determine all the remaining ones so that it is a solution of (3). Substituting (7) into (3) and using the linear independence of $f^\lambda(x)$ and $g^\lambda(x)$ for different λ, we obtain

$$A(\lambda)\xi_{\lambda-r}+C(\lambda)\xi_\lambda=0, \tag{8}$$

$$B(\lambda)\eta_{\lambda-r}+C(\lambda)\eta_\lambda=0. \tag{9}$$

Let r be even. In this case the degrees of $A(\lambda)$ and $B(\lambda)$ are equal to $(2m+r)/2$. Having a root λ_1 $(A(\lambda_1)=0)$, we construct an arithmetic progression with difference r, passing through λ_1. Taking ξ_{λ_1-r} to be an arbitrary constant, from (8) we determine the subsequent values of ξ_λ for all $\lambda=\lambda_1-kr$ $(k=1,2,3,\ldots)$. In this case it is easy to calculate that

$$|\xi_\lambda|\leqslant\frac{A^\lambda}{(\lambda^\lambda)^{(2m+r-2n)/2}},\qquad |\lambda|\to\infty,\quad A=\text{const}. \tag{10}$$

Carrying out this procedure for each root of $A(\lambda)$, we find that equation (8) has $(2m+r)/2$ solutions. In exactly the same way, equation (9) has $(2m+r)/2$ solutions. Hence we obtain that equation (3) has $(2m+r)$ solutions. The convergence of the series (7) in $(S_0^\beta)'$ $(1<\beta<(m+r-n)/r)$ follows from the estimate (10). By the conditions of the theorem these $(2m+r)$ solutions are linearly independent. For odd r the theorem is proved in an analogous way, with the difference that equation (8) has $(2m+r-1)/2$, and equation (9) $(2m+r+1)/2$, linearly independent solutions.

Theorem 2. *Among the linear combinations of the solutions mentioned in Theorem 1, there are $2m$ linearly independent ones, m of which are equal to zero for $x<0$, and the remaining m are equal to zero for $x>0$.*

Proof. We know that equation (3) has exactly r linearly independent solutions concentrated at zero. They are also described by formula (7). Among the remaining ones there are exactly $2m$ linearly independent solutions which are not concentrated at zero. In particular, among them there are $m_+ \leq m$ linearly independent solutions for $x>0$. Denote them by $y_1, y_2, \dots, y_{m_+}$ (below we shall see that $m_+ = m$). The remaining $2m - m_+$ solutions are not concentrated at zero and for $x>0$ are linear combinations of these solutions:

$$y_{m_+ + k} = \sum_{j=1}^{m_+} c_j^k y_j, \qquad k = 1, 2, \dots, (2m - m_+).$$

Obviously,

$$z_k = y_{m_+ + k} - \sum_{j=1}^{m_+} c_j^k y_j, \qquad k = 1, 2, \dots, (2m - m_+)$$

are

are linearly independent solutions of equation (3), equal to zero for $x>0$. Similarly, we obtain that there exist $u_1, u_2, \dots, u_{2m-m_-}$ $(m_- \leq m)$ linearly independent solutions not concentrated at zero and equal to zero for $x<0$. All these solutions, taken together, are linearly independent. Hence it follows that their number $2m - m_+ + 2m - m_- \leq 2m$, $m_+ + m_- \geq 2m$, and since $m_+ \leq m$, $m_- \leq m$, we have $m_+ = m_- = m$, as was required to prove.

Theorem 3. *The dimension of the subspace Ω is equal to $(2m+r)$.*

Proof. Let y be an arbitrary element of Ω. For $x>0$ it coincides with some classical solution, but among the solutions in generalized functions found by us there are m linearly independent ones for $x>0$ and equal to zero for $x<0$. These solutions for $x>0$ are classical, and y is a linear combination of these solutions $z_1, z_2, \dots, z_m$. Similarly, y is a linear combination of the linearly independent solutions $u_1, u_2, \dots, u_m$ for $x<0$. Hence it follows that $u \equiv y - \alpha_1 z_1 - \dots - \alpha_m z_m - \beta_1 u_1 - \beta_2 u_2 - \dots - \beta_m u_m \subset \Omega_0$. Since the dimension of Ω_0 is equal to r, we obtain that y is a linear combination of $(2m+r)$ linearly independent solutions, which proves the theorem.

Remark 2. At the same time we have shown that, under the assumptions of the theorem, every classical solution of equation (3) for $x>0$ (and for $x<0$) is represented by some series (7).

Remark 3. Apparently, Theorem 3 remains valid also in the case when $A(\lambda)$ and $B(\lambda)$ have multiple generalized roots; only in this case one must bring into consideration the associated functions corresponding to $f^{\lambda}(x)$ and $g^{\lambda}(x)$.

Remark 4. If in equation (3) all $b_q = 0$, then Theorem 3 is valid even in K'.

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Note: Figure translations are in progress. See original paper for figures.