ON THE TRANSCENDENCE AND ALGEBRAIC INDEPENDENCE OF VALUES OF \(E\)-FUNCTIONS THAT ARE SOLUTIONS OF A LINEAR DIFFERENTIAL EQUATION OF THIRD ORDER

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ON THE TRANSCENDENCE AND ALGEBRAIC INDEPENDENCE OF VALUES OF E-FUNCTIONS THAT ARE SOLUTIONS OF A LINEAR DIFFERENTIAL EQUATION OF THIRD ORDER

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A. B. Shidlovskii (1) established necessary and sufficient conditions for the algebraic independence of the values, at algebraic points, of a collection of E-functions* that form a solution of a system of linear differential equations of the first order.

As applied to an E-function satisfying a linear differential equation, Shidlovskii's result leads to the following assertion:

Theorem 1 (2). *Let the E-function $f(z)$ be a solution of a linear differential equation of order n*

$$P_n(z)y^{(n)} + P_{n-1}(z)y^{(n-1)} + \cdots + P_0(z)y = Q(z),$$

whose coefficients $Q, P_0, P_1, \dots, P_n$ are polynomials in z, and let α be an algebraic number such that $\alpha P_n(\alpha) \neq 0$. Then, in order that the numbers $f(\alpha), f'(\alpha), \dots, f^{(n-1)}(\alpha)$ be algebraically independent, it is necessary and sufficient that the functions $f(z), f'(z), \dots, f^{(n-1)}(z)$ be algebraically independent over the field of rational functions in z.

Theorem 1 makes it possible to establish the algebraic independence of values of concrete E-functions that are solutions of linear differential equations. But for its application one needs methods that make it possible to establish the algebraic independence of the E-functions under consideration over the field of rational functions.

There is a well-known analytic method (3) applicable to E-functions satisfying a linear differential equation of the second order. By generalizing this method, A. B. Shidlovskii (4) established the algebraic independence of the values of certain special classes of E-functions satisfying linear differential equations of arbitrary order. There are also known arithmetic methods for proving the algebraic independence of functions, developed in the works of C. L. Siegel, A. B. Shidlovskii, and others. However, the indicated methods do not always make it possible to decide the question of the algebraic independence of E-functions. Up to the present time, even a method applicable in the general case to E-functions satisfying linear differential equations of the third order has not been known. Therefore, one of the important problems is the establishment of new methods for proving algebraic independence of functions satisfying differential equations of higher orders.

In the present note a general method, established by the author, is set forth; it is applicable, in particular, to linear homogeneous differential equations of the third order.

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* For the definition of an E-function, see, for example, ([2,3]).

Consider an arbitrary algebraic differential equation of order n

$$P = P(z, y, y', \dots, y^{(n)}) = 0 \quad (n \geqslant 1), \tag{1}$$

where P is a polynomial in $y, y', \dots, y^{(n)}$ with coefficients from the field of rational functions of z.

For $n > 1$, equation (1) will be called **differentially reducible** if at least one solution $y(z) \not\equiv 0$ of this equation satisfies an algebraic differential equation of order less than n; and, for $n = 1$, if it has an algebraic function $y(z) \not\equiv 0$ as a solution. In the contrary case equation (1) is called **differentially irreducible**.

Let, for a fixed natural number k ($k \leqslant n$), $f = f(z, z_1, \dots, z_{k-1})$ be a function of k complex variables, analytic in some domain of the space C^k. We shall denote by G_k the differential operator acting on the function f as follows:

$$G_k f(z, z_1, \dots, z_{k-1}) = \frac{\partial f}{\partial z_{k-1}} f + \frac{\partial f}{\partial z_{k-2}} z_{k-1} + \dots + \frac{\partial f}{\partial z_1} z_2 + \frac{\partial f}{\partial z} \quad (2 \leqslant k \leqslant n),$$

and for $k = 1$, $G_1 f(z) \equiv f'(z)$. By $G_k^t f$ we shall denote the t-fold ($t \geqslant 2$) application of the operator G_k to the function f.

Theorem 2. *Equation (1) is differentially reducible if and only if, for at least one k ($1 \leqslant k \leqslant n$), there exists an algebraic function $f(z, z_1, \dots, z_{k-1})$ satisfying the equation*

$$P(z, z_1, \dots, z_{k-1}, f, G_k f, G_k^2 f, \dots, G_k^{n-k+1} f) = 0. \tag{2}$$

The relation in the left-hand side of equation (2) is obtained by replacing, in the polynomial $P(z, y, y', \dots, y^{(n)})$, the functions $y, y', \dots, y^{(k-2)}$, respectively, by the independent complex variables $z_1, \dots, z_{k-1}$, and the functions $y^{(k-1)}, \dots, y^{(n)}$, respectively, by the functions of these independent variables $f, G_k f, G_k^2 f, \dots, G_k^{n-k+1} f$.

Thus, the question of the differential reducibility of equation (1) is reduced to the problem of finding all algebraic functions $f(z, z_1, \dots, z_{k-1})$ satisfying equations (2), $(1 \leqslant k \leqslant n)$; and if such functions do not exist, then equation (1) is differentially irreducible. This is evidently equivalent to the algebraic independence of the functions $y(z), y'(z), \dots, y^{(n-1)}(z)$, where $y(z) \not\equiv 0$ is an arbitrary solution of equation (1).

To solve the stated problem, we shall establish the possibility of expanding an algebraic function into convergent functional series of the form

$$f(z, z_1, \dots, z_{k-1}) = h_0 z^{\varepsilon_0} + h_1 z^{\varepsilon_1} + \cdots, \tag{3}$$

arranged in powers of z, with coefficients $h_0 = h_0(z_1, \dots, z_{k-1})$, $h_1 = h_1(z_1, \dots, z_{k-1}), \dots$, which are algebraic functions of $z_1, \dots, z_{k-1}$, defined in one and the same domain v of the space C^{k-1}.

In order to verify the existence of such expansions, it is necessary to consider the set Φ of formal series of the form (3), whose exponents satisfy the condition $\varepsilon_0 < \varepsilon_1 < \cdots$ and are rational fractions with a common denominator, while all the coefficients $h_0, h_1, \dots$ are defined and analytic in some domain U of the space C^{k-1}.

Theorem 3. *For every algebraic equation*

$$d_l f^l + d_{l-1} f^{l-1} + \cdots + d_1 f + d_0 = 0, \quad d_l \not\equiv 0, \tag{4}$$

with coefficients d_i $(i = 0, 1, \dots, l)$ belonging to Φ, there exists at least one formal series (3), defined in some domain $v \subseteq U$, which, after substitution for f, turns the left-hand side of (4) identically into zero with respect to $z, z_1, \dots, z_{k-1}$. The number of distinct such series does not exceed the degree l of equation (4).

If, then, instead of the left-hand side of (4), one considers the polynomial $R(z, z_1, \dots, z_{k-1}, f)$, defining the algebraic function $f = f(z, z_1, \dots, z_{k-1})$, then the following is true.

Theorem 4. In the space C^k there exists a simply connected domain B, where the algebraic function f is represented by the convergent series (3), arranged in decreasing rational powers of z: $\varepsilon_0 > \varepsilon_1 > \cdots$ with a common denominator.

We shall apply the indicated theorems to the investigation of the arithmetic nature of the values of the E-functions

$$\left.\begin{aligned} K_{\lambda,\mu}(z) &= \sum_{n=0}^{\infty} \frac{1}{n!(\lambda+1)\cdots(\lambda+n)(\mu+1)\cdots(\mu+n)} \left(\frac{z}{3}\right)^{3n}, \\ K_{\lambda,\mu,\alpha}(z) &= \sum_{n=0}^{\infty} \frac{\alpha(\alpha+1)\cdots(\alpha+n-1)}{n!(\lambda+1)\cdots(\lambda+n)(\mu+1)\cdots(\mu+n)} \left(\frac{z}{2}\right)^{2n}, \\ K_{\lambda,\mu,\alpha,\beta}(z) &= \sum_{n=0}^{\infty} \frac{\alpha(\alpha+1)\cdots(\alpha+n-1)\beta(\beta+1)\cdots(\beta+n-1)}{n!(\lambda+1)\cdots(\lambda+n)(\mu+1)\cdots(\mu+n)} z^n, \end{aligned}\right\} \quad \lambda,\mu \neq -1,-2,\ldots$$

which satisfy respectively the linear differential equations of the 3rd order:

$$y''' + \frac{3(\lambda+\mu+1)}{z} y'' + \frac{(3\lambda+1)(3\mu+1)}{z^2} y' - y = 0, \tag{5}$$

$$y''' - \frac{2\lambda+2\mu+3}{z} y'' - \frac{(2\lambda+1)(2\mu+1)-z^2}{z^2} y' - \frac{2\alpha}{z} y = 0,$$

$$y''' - \frac{\lambda+\mu+3-z}{z} y'' + \frac{(\lambda+1)(\mu+1)-(\alpha+\beta+1)z}{z^2} y' - \frac{\alpha\beta}{z^2} y = 0.$$

Concerning the first of these functions $K_{\lambda,\mu}$, the following holds.

Theorem 5. Let λ, μ be rational numbers satisfying the conditions

$$|\lambda+\mu| \neq n_1; \qquad |\lambda-2\mu| \neq n_2; \qquad |\mu-2\lambda| \neq n_3, \tag{6}$$

where n_1, n_2, n_3 are arbitrary natural numbers.

Then the numbers $K_{\lambda,\mu}(z)$, $K'_{\lambda,\mu}(z)$, $K''_{\lambda,\mu}(z)$ are algebraically independent for any algebraic value $z \neq 0$.*

We outline the proof. By Theorem 1, it is enough to prove that, for the indicated values of λ, μ, the functions $K_{\lambda,\mu}(z)$, $K'_{\lambda,\mu}(z)$, and $K''_{\lambda,\mu}(z)$ are algebraically independent over the field of rational functions in z. This fact is a consequence of the assertion:

For values of λ, μ satisfying the conditions (6), equation (5) is differentially irreducible.

By Theorem 2, to prove this assertion it is enough to establish that every algebraic function $y = f(z)$ satisfying equation (5) is identically zero and, moreover, that there do not exist algebraic functions $f(z, z_1)$, $f(z, z_1, z_2)$ satisfying respectively the equations

$$G_2^2 f + \frac{3(\lambda+\mu+1)}{z} G_2 f + \frac{(3\lambda+1)(3\mu+1)}{z^2} f - z_1 = 0, \tag{7}$$

$$G_3 f + \frac{3(\lambda+\mu+1)}{z} f + \frac{(3\lambda+1)(3\mu+1)}{z^2} z_2 - z_1 = 0. \qquad (8)$$

Substituting into equation (5) the expansion of the function $y = f(z)$ in a series in decreasing powers of z and comparing the coefficients of the highest powers, we obtain $f(z) \equiv 0$.

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* For the values $\lambda, \mu = 0$, this fact was established by A. B. Shidlovskii in work (4).

From the expansion of the algebraic function $f(z, z_1) = h_0(z) z_1^{\varepsilon_0} + h_1(z) z_1^{\varepsilon_1} + \cdots;\ \varepsilon_0 > \varepsilon_1 > \cdots$ (Theorem 4), we obtain that equality (7) is possible for $\lambda + \mu = -n_1 - 1,\ \lambda - 2\mu = -n_2 - 1,\ \mu - 2\lambda = -n_3 - 1$. However, this contradicts conditions (6).

Finally, if the algebraic function $f(z, z_1, z_2)$ satisfies equation (8), then, using the divisibility properties of polynomials in several variables [(3)] and the expansion indicated above, we obtain $\lambda + \mu = n_1 + 1,\ \lambda - 2\mu = n_2 + 1,\ \mu - 2\lambda = n_3 + 1$. The conditions found also contradict (6).

Using analogous techniques for the functions $K_{\lambda,\mu,\alpha}$ and $K_{\lambda,\mu,\alpha,\beta}$, one can prove the following:

Theorem 6. Let λ, μ, α be rational numbers satisfying the conditions $\alpha + \lambda \neq -n_1,\ \alpha + \mu \neq -n_2$, where $n_1, n_2 = 1, 2, \ldots$, and $\alpha - \lambda \neq n_3/2;\ \alpha - \mu \neq n_4/2,\ \alpha \neq n_5/2;\ \alpha + \lambda - \mu \neq (2n_6 - 1)/2;\ \alpha - \lambda + \mu \neq (2n_7 - 1)/2;\ \alpha - \lambda - \mu \neq (2n_8 - 1)/2$, where $n_3, n_4, n_5, n_6, n_7, n_8 = 0, -1, \pm 2, \pm 3, \ldots$. Then the numbers $K_{\lambda,\mu,\alpha}(z),\ K'_{\lambda,\mu,\alpha}(z),\ K''_{\lambda,\mu,\alpha}(z)$ are algebraically independent for every algebraic $z \neq 0$.

Theorem 7. Let $\lambda, \mu, \alpha, \beta$ be rational numbers satisfying the conditions $\alpha + \lambda \neq -n_1,\ \beta + \lambda \neq -n_2,\ \alpha + \mu \neq -n_3,\ \beta + \mu \neq -n_4$, where $n_1, n_2, n_3, n_4 = 1, 2, \ldots;\ \alpha \neq n_5,\ \alpha - \lambda \neq n_6,\ \alpha - \mu \neq n_7,\ \beta \neq n_8,\ \beta - \lambda \neq n_9,\ \beta - \mu \neq n_{10}$, where $n_5, n_6, n_7, n_8, n_9, n_{10} = 0, \pm 1, \pm 2, \ldots;\ \alpha + \beta - \mu \neq n_{11},\ \alpha + \beta - \lambda \neq n_{12},\ \alpha + \beta - \lambda - \mu \neq n_{13}$, where $n_{11}, n_{12}, n_{13} = 0, -1, \pm 2, \pm 3, \ldots$. Then the numbers $K_{\lambda,\mu,\alpha,\beta}(z),\ K'_{\lambda,\mu,\alpha,\beta}(z),\ K''_{\lambda,\mu,\alpha,\beta}(z)$ are algebraically independent for every algebraic $z \neq 0$.

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REFERENCES CITED

[1] A. B. Shidlovskii, DAN, **100**, No. 2 (1955).
[2] A. B. Shidlovskii, Izv. AN SSSR, Ser. Mat., **23**, No. 1 (1959).
[3] C. Siegel, Abh. Preuss. Acad. Wiss., No. 1, 70 (1929-1930).
[4] A. B. Shidlovskii, DAN, **108**, No. 3 (1956).

Note: Figure translations are in progress. See original paper for figures.