# ON A GENERAL THEOREM ON THE ALGEBRAIC INDEPENDENCE OF VALUES OF \(E\)-FUNCTIONS

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## Abstract

## Full Text

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# ON A GENERAL THEOREM ON THE ALGEBRAIC INDEPENDENCE OF VALUES OF $E$-FUNCTIONS

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In the author's paper ([1]) a detailed proof was given of the general theorem on the transcendence and algebraic independence of the values of a set of $E$-functions (for the definition of an $E$-function see ([1,3,4])) satisfying linear differential equations with coefficients from the field of rational functions, first published without proof in ([2]). The proof of this theorem is a generalization of K. Siegel' s method, first published in ([3]) and also presented by him in the form of a general theorem in ([4]). Both in Siegel' s method and in the proof of the general theorem of paper ([1]), one of the basic propositions is a lemma on the rank of a set of values of the functions under consideration with respect to a certain algebraic field $K$ (Lemma 7 in ([4]) and Lemma 9 in ([1]), proved under substantially different assumptions), from which all the arithmetic results proved there follow easily.

It turns out that this lemma can be strengthened. In the present paper we shall make changes in the proof of the general theorem of paper ([1]) in order to prove Lemma 9 in a strengthened form, and then obtain some new arithmetic consequences from this lemma.

Let us replace Lemma 1 of paper ([1]) by the following proposition:

**Lemma 1** (see ([4])). Let

$$f_1(z), \dots, f_m(z) \tag{1}$$

be entire functions; let $\varepsilon$ be any real number, $0 < \varepsilon < 1$; and let $n$ be any natural number. Then there exist $m$ polynomials

$$P_k(z) = \sum_{s=0}^{2n-1} b_{k,s} z^s, \quad k = 1, \dots, m, \tag{2}$$

whose coefficients $b_{k,s}$ are not all zero and whose degrees do not exceed $2n-1$, with the following properties:

1) The linear form

$$R = \sum_{k=1}^{m} P_k(z) f_k(z) = \sum_{\nu=0}^{\infty} a_\nu \frac{z^\nu}{\nu!} \tag{3}$$

has at $z = 0$ a zero of order at least $2mn - [\varepsilon n] - 1$*, so that

$$a_\nu = 0, \quad \nu = 0, 1, \dots, 2mn - [\varepsilon n] - 2. \tag{4}$$

If, moreover, the functions (1) are $E$-functions, the coefficients of whose power series belong to the algebraic field $K$, then:

2) The coefficients of the polynomials (2), $b_{k,s}$, are integers of the field $K$, satisfying the condition

$$|\overline{b_{k,s}}| = O\left(n^{(2+\varepsilon)n}\right), \quad k = 1, \dots, m;\ s = 0, 1, \dots, 2n-1, \tag{5}$$

uniformly in $k$ and $s$;

* $[x]$ denotes the integer part of the number $x$.

3) The coefficients $a_\nu$ of the linear form $R$ (3), for sufficiently large $n$, satisfy the condition

$$|a_\nu| = \nu^\nu O(n^{2n}), \qquad \nu \geq 2mn - [\varepsilon n] - 1, \tag{6}$$

uniformly in $\nu$.

**Proof.** Put

$$f_k(z) = \sum_{\nu=0}^{\infty} c_{k,\nu} \frac{z^\nu}{\nu!}, \qquad P_k(z) = (2n-1)! \sum_{\nu=0}^{2n-1} g_{k,\nu} \frac{z^\nu}{\nu!}, \qquad k = 1, \dots, m, \tag{7}$$

with integral $g_{k,\nu}$ in $K$. Then $P_k(z)$ is a polynomial in $z$ of degree not exceeding $2n-1$ with integral coefficients from the field $K$.

Computing the coefficients $a_\nu$ in the linear form (3), we obtain

$$P_k(z) f_k(z) = (2n-1) \sum_{\nu=0}^{\infty} d_{k,\nu} \frac{z^\nu}{\nu!}, \qquad d_{k,\nu} = \sum_{\rho=0}^{2n-1} \binom{\nu}{\rho} g_{k,\rho} c_{k,\nu-\rho}, \tag{8}$$

since $g_{k,\rho} = 0$ for $\rho > 2n-1$, and

$$a_\nu = (2n-1)! \sum_{k=1}^{m} d_{k,\nu}, \qquad \nu = 0, 1, \dots \tag{9}$$

From condition (4) we obtain $2mn - [\varepsilon n] - 1$ linear homogeneous equations for determining the $2mn$ unknown coefficients $g_{k,\rho}$, $k = 1, \dots, m$, $\rho = 0, 1, \dots, 2n-1$. Choose a natural number $q_n$ so that all the numbers $q_n c_{k,\nu}$, $k = 1, \dots, m$, $\nu = 0, 1, \dots, 2mn - [\varepsilon n] - 2$, are integral numbers of the field $K$. From the definition of $E$-functions it follows that $q_n = O(n^{\varepsilon n})$. Multiply both sides of the equation $a_\nu = 0$ by $q_n/(2n-1)!$. Then in the resulting equations the coefficients at $g_{k,\rho}$ will be the numbers

$$q_n \binom{\nu}{\rho} c_{k,\nu-\rho},$$

which are integral numbers of the field $K$, and

$$\left| q_n \binom{\nu}{\rho} c_{k,\nu-\rho} \right| = O(n^{\varepsilon n}), \tag{10}$$

since $\binom{\nu}{\rho} \le 2^\nu$, and, by the definition of $E$-functions, $\overline{|c_{k,\nu}|} = O(\nu^{\varepsilon\nu})$, and $\nu < 2mn - [\varepsilon n] - 1$. Using Lemma 2 of the paper [4] with the values $p = 2mn - [\varepsilon n] - 1$, $q = 2mn$, $A = O(n^{\varepsilon n})$, we obtain that the $g_{k,\rho}$ are integral numbers of the field $K$, not all zero, and such that

$$\overline{|g_{k,\rho}|} = O(n^{\varepsilon n}), \qquad k = 1, \dots, m; \ \rho = 0, 1, \dots, 2n-1, \tag{11}$$

uniformly in $k$ and $\rho$, since

$$p/(q-p) = (2mn - [\varepsilon n] - 1)/([\varepsilon n] + 1) < 2m/\varepsilon.$$

Then the equalities (5) follow from (7) and (11), and the equalities (6) from (8), (9), (10), and (11) and the estimate $(2n-1)! = O(n^{2n})$.

Let us change the definition of the number $t$ in formula (42) of the paper [1], putting $t = [\varepsilon n] + p + qm(m-1)/2$, where $0 < \varepsilon < 1$, and $n, p, q$ have the same meaning as in (42). Then, with the aid of the modified Lemma 1, Lemma 6 from [1] is also established in a slightly modified form. The proof proceeds by literal repetition with the introduction of the formal changes supplied by Lemma 1.

**Lemma 6** (see [1]). Suppose a collection of integral functions (1) is a solution of the system of linear homogeneous differential equations

$$y_k' = \sum_{i=1}^{m} Q_{k,i}(z), \qquad k = 1, \dots, m, \tag{12}$$

all of whose coefficients $Q_{k,i}(z)$ are rational functions of $z$, and is linearly independent over the field of rational functions. Suppose, furthermore, that the linear approximating form

$$R_1 = \sum_{i=1}^{m} P_{1,i} y_i$$

has been constructed by Lemma 1 for some value of $n$ and arbitrary $\varepsilon$, $0 < \varepsilon < 1$, with the subsequent replacement of the functions (1) by an arbitrary solution $y_1, \dots, y_m$ of the system (12). Then there exists a natural number $n_0$ such that, for every $n \geq n_0$, the determinant $\Delta = \Delta(z) = |P_{k,i}|$, $k, i = 1, \dots, m$, of the system of linear forms

$$R_k = \sum_{i=1}^{m} P_{k,i} y_i, \quad k = 1, \dots, m,$$

is not identically equal to zero in $z$ and has the form

$$\Delta(z) = z^{2mn-[\varepsilon n]-m-p} \Delta_1(z), \quad \Delta_1(z) \neq 0, \quad n \geq n_0,$$

where $\Delta_1(z)$ is a polynomial in $z$ of degree $r_1$, and $r_1$ satisfies the inequalities $0 < r_1 < t$.

Lemma 7 of [1], thanks to Lemma 6, remains valid for the new value of $t$. The proof is unchanged.

Lemma 8, formulated in [1], whose proof is given in [4] (Lemma 6), is retained in the same formulation (for $0 < \varepsilon < 1$), but with changes in the assertion in equalities (81) and (82), which respectively take the form

$$|R_k(\alpha)| = O\left(n^{-(2m-2-\varepsilon)n}\right), \qquad \overline{|P_{k,i}(\alpha)|} = O\left(n^{(2+\varepsilon)n}\right), \quad i = 1, \dots, m.$$

The proof proceeds analogously to that given in [4]; one need only note that, by Lemma 1, in equality (82) the expansion of $\widetilde{R}$ begins not with degree $(2m-1)n$, but with degree $2mn - [\varepsilon n] - 1$, and at the end of the proof, in the estimates, instead of the inequality $k \leq m + t = n + O(1)$, one must use the inequality $k \leq m + t - \varepsilon n + O(1)$.

Lemma 9 of [1] is now proved in the following formulation:

**Lemma 9** (see [1]). Let a collection of $E$-functions (1) be a solution of the system of linear homogeneous differential equations (12), all of whose coefficients $Q_{k,i}$ are rational functions, and be linearly independent over the field of rational functions; and let the coefficients of the power series of these functions and the number $\alpha$ belong to an algebraic field $K$ of degree $h$ over the field of rational numbers, with $\alpha T(\alpha) \neq 0$.* Then the rank of the $m$ numbers $f_1(\alpha), \dots, f_m(\alpha)$ relative to the field $K$ is not less than $m/h$; and if $K$ is an imaginary quadratic field, then the rank of these numbers is equal to $m$.

The proof of the lemma proceeds word for word as in [1], except that instead of estimates (89) and (90) one must take the modified estimates from (81) and (82), and then estimates (91), (92), (94), and (95) take, respectively, the form

$$\overline{|\delta_{k,i}|} = O\left(n^{(2+\varepsilon)(r-1)n}\right), \qquad \overline{|\delta|} = O\left(n^{(2+\varepsilon)rn}\right),$$

$$\delta = O\left(n^{[(2+\varepsilon)r-2m]n}\right), \qquad N(\delta) = O\left(n^{[(2+\varepsilon)rh-2m]n}\right),$$

whence, in view of (96), we obtain the inequality $(2+\varepsilon)rh - 2m > 0$, and, by virtue of the arbitrariness of $\varepsilon$, the inequality $r \geq m/h$. The result in the case of an imaginary quadratic field $K$ follows from the fact that $\delta$ and its conjugate in the field $K$ have one and the same modulus.

With the aid of Lemma 9, one analogously establishes the lemma refining Lemma 2 of [5], with the replacement in its assertion of $r/2h$ by $r/h$.

**Theorem.** Let $K$ be either the field of rational numbers or an imaginary quadratic algebraic field over the field of rational numbers; let a collection of $E$-functions (1), with coefficients of the power series from the field $K$, constitute a solution of the system of linear differential equations

$$y_k' = Q_{k,0} + \sum_{i=1}^{m} Q_{k,i} y_i, \quad i = 1, \dots, m, \tag{13}$$

* $T(z)$ is a polynomial which is the common least denominator of all the functions $Q_{k,i}$.

all coefficients of which $Q_{k,i}$ are rational functions of $z$, and $\alpha$ is any number in the field $K$ different from zero and from the poles of all the functions $Q_{k,i}$.

Then the following assertions hold:

1) In order that the numbers

$$f_1(\alpha), \dots, f_m(\alpha) \tag{14}$$

and the number 1 be linearly independent over the field $K$, it is necessary and sufficient that the functions (1), together with $f(z) \equiv 1$, be linearly independent over the field of rational functions. If the latter condition is fulfilled, then, in particular, each of the numbers (14) does not belong to the field $K$ and is therefore irrational; and then none of these functions has zeros in $K$ different from zero and from the poles of all the functions $Q_{k,i}$.

2) In order that the numbers (14) should not be connected by an algebraic equation with coefficients from $K$ of degree $k$, $k \geq 1$, it is necessary and sufficient that the functions (1) should not be connected by an algebraic equation with coefficients—rational functions—of degree $k$.

3) If the system (13) is homogeneous, then in order that the numbers (14) should not be connected by a homogeneous algebraic equation with coefficients from the field $K$, of degree $k$, $k \geq 1$, it is necessary and sufficient that the functions (1) should not be connected by a homogeneous algebraic equation with coefficients—rational functions—of degree $k$. In particular, if the latter condition is fulfilled for $k = 1$, then none of these functions has zeros in $K$ different from zero and from the poles of all the functions $Q_{k,i}$.

The proof of the theorem follows from Lemma 9 just as simply and analogously as the main theorem and Theorem 3 of paper [(1)]; moreover, the assertions of those theorems, in the case of the field $K$ under consideration, are limiting cases of assertions 2 and 3, when the functions under consideration are not connected by any corresponding algebraic equations over the field of rational functions.

The assertions of this theorem are not difficult to reformulate for the case of an $E$-function that is a solution of a linear differential equation with polynomial coefficients, and of the sequence of its successive derivatives, as is done in Theorem 1 of paper [(1)].

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## REFERENCES

[1] A. B. Shidlovskii, *Izv. AN SSSR, Ser. Mat.*, **23**, No. 1, 35 (1959).
[2] A. B. Shidlovskii, *DAN*, **100**, No. 2 (1955).
[3] C. L. Siegel, *Abh. Preuss. Acad. Niss.*, No. 1 (1929–1930).
[4] C. L. Siegel, *Transcendental Numbers*, Princeton, 1949.
[5] A. B. Shidlovskii, *Izv. AN SSSR, Ser. Mat.*, No. 6, 877 (1962).

*Note: Figure translations are in progress. See original paper for figures.*