SOME CONDITIONS FOR UNIQUENESS OF THE SOLUTION OF INVERSE PROBLEMS OF POTENTIAL THEORY

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Abstract

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SOME CONDITIONS FOR UNIQUENESS OF THE SOLUTION OF INVERSE PROBLEMS OF POTENTIAL THEORY

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Two inverse problems for the metaharmonic potential are considered.

Problem 1. Determine the density of a given body from its exterior potential.

Problem 2. Determine the shape of a body of given density from its exterior, or interior, potential.

These problems, whose solution, generally speaking, is nonunique, are ill-posed in the classical sense. The question of a correct formulation and methods for solving problems of this kind, many of which reduce to linear and nonlinear integral equations of the first kind, is treated in papers ([2],[4],[8]), etc. In the present article the question of uniqueness of the solution of the indicated inverse problems is investigated. In particular, for the case $\chi = 0$ the results obtained are also new for the Newtonian potential.

1°. Let A_α ($\alpha = 1, 2$) be open bounded subsets of the space E^n ($n \geq 2$). Denote the metaharmonic potential ($\chi \geq 0$) ([5]) of the set A_α with density $\mu_\alpha(y)$ ($\alpha = 1, 2$), nonzero almost everywhere for points $y \in A_\alpha$, by

$$V(x; A_\alpha, \mu_\alpha) = \int_{A_\alpha} \mu_\alpha(y) K(x, y)\, dy, \tag{1}$$

where $K(x, y)$ is the fundamental solution of the metaharmonic equation

$$\Delta u - \chi^2 u = 0 \qquad (\chi = \text{const} \geq 0). \tag{2}$$

2°. Consider problem 1. In the author' s paper ([5]) the question of uniqueness of the solution of problem 1 was investigated for certain classes of densities $\mu_\alpha(y)$. Here uniqueness theorems are given that strengthen the results of ([5]). Let $A = A_1 = A_2$. Denote by

$$V(x; A, \mu_\alpha) = V(x; A_\alpha, \mu_\alpha) \qquad (\alpha = 1, 2)$$

the potential of the given set A with density $\mu_\alpha(y)$, $y \in A$. Consider the functions $\mu_\alpha(y)$ $(\alpha = 1, 2)$ satisfying the conditions:

$$\mu_\alpha(y) = \beta(t)\delta_\alpha(y'), \qquad (y', t) = y, \quad y' = (y_1, \dots, y_{n-1}), \quad t = y_n, \tag{3}$$

$$\delta_\alpha(y') \text{ are bounded summable functions of the points } y' \in A' \tag{4}$$

$$(A' \text{ is the projection of the set } A \text{ onto the plane } t = 0);$$

$$\beta(t) \text{ is a positive monotone function, } \beta(t) \in C^1(\overline{A}). \tag{5}$$

Theorem 1 (case $\chi \geq 0$). *If, for functions $\mu_\alpha(y)$ $(\alpha = 1, 2)$ satisfying conditions (3)–(5), the exterior potentials of the given set A coincide, i.e.*

$$V(x; A, \mu_1) = V(x; A, \mu_2) \qquad \text{for } x \in E^n \setminus \overline{A},$$

then

$$\mu_1(y) = \mu_2(y) \qquad \text{for } y \in A.$$

Let now the functions $\mu_\alpha(y)$ $(\alpha = 1, 2)$ satisfy the conditions:

$$\mu_\alpha(y) = \beta(\rho)\delta_\alpha(\theta) \quad (\alpha = 1, 2), \tag{6}$$

where (ρ, θ) are the spherical coordinates of the point y of the space E^n;

$$\delta_\alpha(\theta) \quad (\alpha = 1, 2)$$

are bounded summable functions on the unit sphere Ω, over which the point θ ranges;

(7)

$$\beta(\rho) > 0, \quad \psi(\rho) = \rho^n \beta$$

is a monotone function of ρ, $\beta \in C^1(\bar{A})$.

(8)

Theorem 2 (the case $\chi = 0$). If, for functions $\mu_\alpha(y)$ $(\alpha = 1, 2)$ satisfying conditions (6)–(8), the exterior potentials of the given set A coincide, i.e.

$$V(x; A, \mu_1) = V(x; A, \mu_2) \quad \text{for } x \in E^n \setminus \bar{A},$$

then

$$\mu_1(y) = \mu_2(y) \quad \text{for } y \in A.$$

3°. We investigate the uniqueness of the solution of problem 2 under equality of the exterior potentials. In the work ([5]) and others, this problem was investigated for a given positive density. Here we give uniqueness theorems for the solution of problem 2 for given variable densities of arbitrary sign.

Let $\mu(y) = \mu_1(y) = \mu_2(y)$ (see 1°). Denote by

$$V(x; A_\alpha, \mu) = V(x; A_\alpha, \mu_\alpha) \quad (\alpha = 1, 2)$$

the potential of the given density $\mu(y)$ of the set A_α $(\alpha = 1, 2)$.

We shall say that the sets A_α $(\alpha = 1, 2)$ possess a **common median plane** ([7]) if there exists a plane H, $t = H = \text{const}$, called the median plane, such that every straight line parallel to the direction of the axis *ot* intersects Γ_α, the boundary of A_α $(\alpha = 1, 2)$, in no more than two points (or two segments) lying on different sides of the plane. Let the function $\mu(y)$ satisfy the conditions:

$$\mu(y) = \beta(t)\delta(y'), \tag{9}$$

where $\delta(y')$ is a summable bounded function of the points $y' \in A_\alpha$;

$$\beta(t) > 0; \quad \frac{\partial \beta}{\partial t} \geq 0 \text{ for } t > H; \quad \frac{\partial \beta}{\partial t} \leq 0 \text{ for } t \leq H. \tag{10}$$

Theorem 3 (the case $\chi > 0$). If the potentials of the sets A_α $(\alpha = 1, 2)$, possessing a median plane, with a given density $\mu(y)$ of the form (9)–(10), satisfy the condition

$$V(x; A_1, \mu) = V(x; A_2, \mu) \quad \text{for } x \in E^n \setminus (\bar{A}_1 \cup \bar{A}_2),$$

then

$$A_1 = A_2.$$

Theorem 4 (the case $\chi \geq 0$). If the potentials $V(x; A_\alpha, \mu)$ $(\alpha = 1, 2)$ are such that:

1) every straight line parallel to the axis ot intersects Γ_1 and Γ_2 in no more than two points (or two segments);
2) the density $\mu(y)$ has the form (9), with $\beta(t) \equiv \text{const}$;
3) $V(x; A_1, \mu) = V(x; A_2, \mu)$ for $x \in E^n \setminus (\bar{A}_1 \cup \bar{A}_2)$,

then

$$A_1 = A_2.$$

Remark 1. A more general assertion is true, namely: theorem 3 of the author' s work ([5]) holds if the density $\mu(y)$, instead of condition 2 of theorem 3 of work ([5]), satisfies condition (9), where $\beta(t) \equiv \text{const}$, $ot = \vec{q}$.

Let the function $\mu(y)$ have the form

$$\mu(y) = \beta(\rho)\delta(\theta), \tag{11}$$

where $\delta(\theta)$ is a bounded summable function of the points $\theta \in \Omega$,

$$\beta(\rho) > 0, \quad \frac{\partial}{\partial \rho}(\rho^n \beta) \geq 0 \quad \text{for } y \in \bar{A}_\alpha. \tag{12}$$

Let T_α $(\alpha = 1, 2)$ be finite domains of the space E^n.

Theorem 5 (case $\chi = 0$). *If there exists at least one point $O \in \overline{T}_1 \cap \overline{T}_2$ such that:*

1) *the set $\overline{T}_1 \cap \overline{T}_2$ is star-shaped with respect to the point O, and the set $E^n \setminus (\overline{T}_1 \cup \overline{T}_2)$ consists of one component;*
2) *for the functions $\mu(y)$ conditions (11)–(12) are satisfied when the origin of coordinates is chosen at the point O;*
3) *for the domains T_α $(\alpha = 1, 2)$, for the given density $\mu(y)$, the equality holds*

$$V(x; T_1, \mu) = V(x; T_2, \mu) \qquad \text{for } x \in E^n \setminus (\overline{T}_1 \cup \overline{T}_2),$$

then

$$T_1 = T_2.$$

Remark 2. If the potentials $V(x; A_\alpha, \mu)$ $(\alpha = 1, 2)$ with density $\mu(y)$ of the form (11)–(12) satisfy the condition

$$V(x; A_1, \mu) = V(x; A_2, \mu)$$

for $x \in E^n \setminus (\overline{A}_1 \cup \overline{A}_2)$, then $(\overline{A}_1 \cap \overline{A}_2) \neq 0$.

4°. Let us investigate the uniqueness of the solution of problem 2 for contact bodies under equality of the external potentials. For the Newtonian potential ($\chi = 0$), the external contact problem was studied for constant density in papers ([1,10]) under certain restrictions on the sets A_α. Methods for solving the external contact problem were proposed in paper ([9]) and in works on geophysics. Below we give uniqueness theorems for the solution of the external contact problem for given variable densities.

Let A_α ($\alpha = 1, 2$) be finite open sets with boundaries Γ_α. We shall say that A_α has a contact plane if

$$\Gamma_\alpha = H_\alpha \cup \hat{\Gamma}_\alpha,$$

where H_α is part of the plane $t = H = \text{const}$, called the contact plane, the sets $\hat{\Gamma}_1$ and $\hat{\Gamma}_2$ lie on one side of the plane, and every straight line perpendicular to the contact plane intersects $\hat{\Gamma}_\alpha$ in not more than one point (or in one segment).

Let the density $\mu(y)$ have the form

$$\mu(y) = \beta(t)\delta(y'), \tag{13}$$

where $\beta(t)$ satisfies conditions (5), and $\delta(y') = \delta_1(y') = \delta_2(y')$ satisfies conditions (6).

Theorem 6 (case $\chi \geq 0$). *If the potentials of the sets A_α ($\alpha = 1, 2$), having a contact plane, with density $\mu(y)$ of the form (13) satisfy the condition*

$$V(x; A_1, \mu) = V(x; A_2, \mu) \qquad \text{for } x \in E^n \setminus (\overline{A}_1 \cup \overline{A}_2),$$

then

$$A_1 = A_2.$$

We shall say that the sets A_α ($\alpha = 1, 2$) have a **contact spherical surface** if

$$\Gamma_\alpha = R_\alpha \cup \Gamma'_\alpha,$$

where R_α is part of the spherical surface $\rho = R = \text{const}$, called the contact spherical surface, while Γ'_1 and Γ'_2 lie on one side of the contact surface and every ray issuing from the origin intersects Γ'_α in not more than one point (or in one segment).

Let the density $\mu(y)$ have the form

$$\mu(y) = \beta(\rho)\delta(\theta), \tag{14}$$

where $\beta(\rho)$ satisfies conditions (8), and $\delta(\theta) = \delta_1 = \delta_2$ satisfies conditions (7).

Theorem 7 (case $\chi = 0$). *If the potentials of the sets A_α ($\alpha = 1, 2$), having a contact spherical surface, with density*

of the form (14) satisfy the conditions

$$V(x; A_1, \mu) = V(x; A_2, \mu) \quad \text{for } x \in E^n \setminus (\overline{A}_1 \cup \overline{A}_2),$$

then

$$A_1 = A_2.$$

We shall call the sets A_1 and A_2 *externally contact* if each component of the set $A_0 = A_1 \cap A_2$ has a common boundary of the n-dimensional space with one of the components of the set $E^n \setminus (\overline{A}_1 \cup \overline{A}_2)$; it is assumed that $A_0 \neq 0$.

Theorem 8 (case $\chi \geq 0$). If, for externally contact sets A_α ($\alpha = 1, 2$), with a positive summable bounded density $\mu(y)$, the equality

$$V(x; A_1, \mu) = V(x; A_2, \mu) \quad \text{for } x \in E^n \setminus (\overline{A}_1 \cup \overline{A}_2),$$

holds, then

$$A_1 = A_2.$$

Remark 3. If the potentials $V(x, A_\alpha, \mu)$ with positive summable density $\mu(y)$ satisfy the condition $V(x; A_1, \mu) = V(x; A_2, \mu)$ ($\chi \geq 0$) for $x \in E^n \setminus (\overline{A}_1 \cup \overline{A}_2)$, then $(\overline{A}_1 \cap \overline{A}_2) \neq 0$.

5°. We investigate the uniqueness of the solution of problem 2 when the internal potentials of contact bodies are equal. In the classical theory of the Newtonian potential, internal inverse problems of potential theory were considered in connection with problems in the theory of the figure of the Earth and celestial mechanics (see ([3],[6])). For the internal potential, Dive proved uniqueness of the solution of problem 2 in the class of convex domains with positive density. Below we give uniqueness theorems for the solution of the internal contact problem for prescribed variable densities.

Theorem 9 (case $\chi \geq 0$). If the potentials of the sets A_α, having a contact plane, with density $\mu(y)$ of the form (13) satisfy the condition

$$V(x; A_1, \mu) = V(x; A_2, \mu) \quad \text{for } x \in (A_1 \cap A_2),$$

then

$$A_1 = A_2.$$

Theorem 10 (case $\chi = 0$). If the potentials of the sets A_α, having a contact spherical surface, with density $\mu(y)$ of the form (14) satisfy the condition

$$V(x; A_1, \mu) = V(x; A_2, \mu) \quad \text{for } x \in (A_1 \cap A_2),$$

then

$$A_1 = A_2.$$

We shall call the sets A_1 and A_2 *internally contact* if each component of the set $E^n \setminus (\overline{A}_1 \cup \overline{A}_2)$ has a common boundary of the n-dimensional space with one of the components of the set $A_0 = A_1 \cap A_2$, $A_0 \neq 0$.

Theorem 11 (case $\chi \geq 0$). If, for internally contact sets A_α ($\alpha = 1, 2$), with a positive summable bounded density, the equality

$$V(x; A_1, \mu) = V(x; A_2, \mu) \quad \text{for } x \in (A_1 \cap A_2),$$

holds, then

$$A_1 = A_2.$$

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Note: Figure translations are in progress. See original paper for figures.