AN ANALOGUE OF ONE THEOREM OF BASS FOR MODULES OF REPRESENTATIONS OF NONCOMMUTATIVE ORDERS

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Abstract

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AN ANALOGUE OF ONE THEOREM OF BASS FOR MODULES OF REPRESENTATIONS OF NONCOMMUTATIVE ORDERS

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In Bass' s paper ([1]) the following theorem was proved. Let Λ be a Noetherian integral domain whose integral closure is a finitely generated Λ-module. Then, in order that every finitely generated torsion-free Λ-module decompose into a direct sum of ideals of the ring Λ, it is necessary and sufficient that every ideal of the ring Λ have two generators. A similar sufficient condition was obtained independently in ([2]) for orders of commutative semisimple separable algebras.

In the present paper it will be shown that the sufficiency (but not the necessity) of Bass' s condition holds for modules of representations of orders of semisimple separable algebras. Here the results and terminology of D. K. Faddeev ([3]) will be used essentially.

Let o be a Dedekind ring, k its field of fractions, Λ an o-order in a finite-dimensional semisimple separable algebra $\widetilde{\Lambda}$ over k.

Theorem. *In order that every finitely generated Λ-module without o-torsion decompose into a direct sum of ideals, it is sufficient that every right ideal of Λ have two generators.*

Denote by $\mu_l(M)$ ($\mu_r(M)$) the minimal number of generators of a left (right) module M; $\Lambda^* = \mathrm{Hom}(\Lambda, o)$ ([3]).

Proposition 1. *If o is a complete local ring and for every order Λ' containing Λ and considered as a Λ-module, $\mu(\Lambda') \leq 2$, then $\mu_l(\Lambda^*) = \mu_r(\Lambda^*) = 1$.*

It will be enough for us to prove that $\mu_l(\Lambda^*) = 1$, since from ([3]) it follows that in this case also $\mu_r(\Lambda^*) = 1$.

It is clear that Λ will be a semiprimary ([4]) ring, i.e., the factor ring $\overline{\Lambda}$ of the ring Λ by its Jacobson radical R will satisfy the chain conditions on ideals and, hence, will be semisimple in the classical sense. Let $W_1, \dots, W_k$ be the maximal two-sided ideals of Λ corresponding to the decomposition of $\overline{\Lambda}$ into a direct sum of rings; let $U_1, \dots, U_k$ be the corresponding simple right Λ-modules.

From Nakayama' s lemma and from the fact that $\Lambda^*/R\Lambda^*$ is a module over the ring $\overline{\Lambda}$, semisimple in the classical sense, it follows that

$$\mu_l(\Lambda^*) = \mu_l(\Lambda^*/R\Lambda^*) = \max_i \mu_l(\Lambda^*/W_i\Lambda^*);$$

$\Lambda^*/W_i\Lambda^*$ is a module over the simple ring Λ/W_i. Note that if M is a module over a simple ring in the classical sense, i.e., a full matrix ring of order n over some skew field, then the minimal number of generators $\mu(M)$ of the module M is completely determined by the length of a composition series $l(M)$ of the module M. Namely, $\mu(M) = 1 \leftrightarrow l(M) \leq n$; $\mu(M) = 2 \leftrightarrow n < l(M) \leq 2n$, and so on. Using this consideration and the results of ([3]), we obtain

$$\mu_l(\Lambda^*/W_i\Lambda^*) = \mu_r((\Lambda^*/W_i\Lambda^*)^*) = \mu_r(\Lambda : W_i/\Lambda).$$

Put

$$T_i = \Lambda : W_i = \{x \mid x \in \widetilde{\Lambda},\ xW_i \in \Lambda\}.$$

In order to prove Proposition 1, we must show that $\mu_r(T_i/\Lambda) = 1,\ i = 1, \dots, k$.

Suppose first that W_i is not a principal right ideal. We shall show that in this case

$$T_i = \Lambda_l(W_i) = \{x \mid x \in \hat{\Lambda},\ xW_i \subseteq W_i\}. \tag{3}$$

It can be shown that $T_i = \sum_\alpha B^i_\alpha$, where B^i_α is a right Λ-module, $B^i_\alpha \supset \Lambda$, and $B^i_\alpha : \Lambda \simeq U_i$. Thus it is enough to prove that $B^i_\alpha W_i \subseteq W_i$. Since $B^i_\alpha \supset \Lambda$, we have $B^i_\alpha W_i \supseteq W_i$. Suppose that $B^i_\alpha W_i \supset W_i$. We note that for $i \neq j$, $B^i_\alpha W_j \supset W_j$. Therefore

$$l(B^i_\alpha/B^i_\alpha W_j) \leq l(\Lambda/W_j), \qquad 1 \leq j \leq k.$$

But since $B^i_\alpha/B^i_\alpha W_j$ is a module over the simple ring Λ/W_j in the classical sense, it follows from the inequality for the lengths of composition series that

$$\mu(B^i_\alpha/B^i_\alpha W_j) \leq \mu(\Lambda/W_j) = 1,$$

i.e. $B^i_\alpha/B^i_\alpha W_j$ is a cyclic module. Using Schanuel' s theorem ([6]) and the uniqueness of the decomposition ([7]), one can show that in this case all maximal ideals of the ring Λ whose factor-module is isomorphic to u_i are principal, and hence W_i is principal, which contradicts our assumption.

Thus, if W_i is not a principal right ideal, then $T_i = \Lambda_l(W_i)$. Since

$$\mu_r(\Lambda_l(W_i)) \leq 2,$$

we have

$$l(\Lambda_l(W_i)/\Lambda_l(W_i)W_i) \leq 2l(\Lambda/W_i),$$

but $\Lambda_l(W_i)W_i = W_i$, whence

$$l(\Lambda_l(W_i)/\Lambda_l(W_i)W_i) = l(\Lambda_l(W_i)/\Lambda) + l(\Lambda(W_i)),$$

$$l(\Lambda_l(W_i)/\Lambda) \leq l(\Lambda/W_i),$$

and hence $\mu_r(T_i/\Lambda) = 1$. If, however, $\mu_r(T_i/\Lambda) = 1$, then it is not difficult to show that also

$$\mu_l(T_i) = \mu_l(T_i/\Lambda) = \mu_r(T_i/\Lambda) = 1.$$

The proposition is proved.

Let A and B be two modules over an arbitrary ring. Denote by $A \circ B$ the submodule of the module B generated by the images of all homomorphisms from the module A into the module B.

It is not difficult to see that

$$A^{(m)} \circ B^{(n)} = (A \circ B)^{(n)},$$

where $M^{(n)}$ is the direct sum of n copies of the module M. We note that if A and B are right ideals of the order Λ, then in the terminology of ([3])

$$A \circ B = (B : A)A.$$

Following ([8]), we shall say that the module A divides the module B if $A \circ B = B$; we shall say that the module A is normally indecomposable if there is no decomposition

$$A = A_1 \oplus A_2$$

such that A_1 divides A_2 and $A_2 \neq \{0\}$.

By Λ_i we shall denote the right multiplier ring ([3]) of a certain finit generated Λ-module M without o-torsion. If

$$\tilde{M} = M \otimes_o k$$

contains in its decomposition all simple modules over the algebra

$$\tilde{\Lambda} = \Lambda \otimes_o k,$$

then $\Lambda_r(M)$ will be an overorder of the order Λ in the algebra $\tilde{\Lambda}$. If, however,

$$\tilde{\Lambda} = \tilde{\Lambda}_1 \oplus \cdots \oplus \tilde{\Lambda}_t$$

is the decomposition of the algebra $\tilde{\Lambda}$ into a direct sum of simple algebras and $\tilde{M}$ contains the simple modules corresponding to the simple components $\tilde{\Lambda}_1, \ldots, \tilde{\Lambda}_s$, $s < t$, then $\Lambda_r(M)$ will be a (ring) direct summand of some overorder of the order Λ, lying in the algebra

$$\tilde{\Lambda}_1 \oplus \cdots \oplus \tilde{\Lambda}_s.$$

Each Λ_i may be regarded as a right Λ-module.

We note that if every right ideal of Λ has two generators, then every right ideal in Λ_i also has two generators.

It is not difficult to verify the validity of the following assertion.

Lemma 1. If $\Lambda_1 \circ \Lambda_2 = \Lambda_2$, then

$$\Lambda_2^* \circ \Lambda_1^* = \Lambda_2^*.$$

We shall say that an o-order Λ **satisfies condition** $(*)$ if o is a complete local ring and for every Λ_i

$$\Lambda_i^* \simeq \Lambda_i,$$

i.e.

$$\mu_r(\Lambda_i^*) = \mu_l(\Lambda^*) = 1.$$

Proposition 2. Let Λ be an order satisfying condition $(*)$. Then, if A is a finit generated normally indecomposable Λ-module without o-torsion,

$$A = A_1 \oplus \cdots \oplus A_s,$$

where the A_i are indecomposable, then

$$\Lambda_r(A) = A_1^{(m_1)} \oplus \cdots \oplus A_s^{(m_s)}.$$

Indeed, from ([3]) it follows that A divides

$$\Lambda_r^*(A) = \Lambda_r(A).$$

On the other hand, obviously, $\Lambda_r(A)$ divides A. Thus, there exist ([8]) two exact sequences

$$A^{(m)} \to \Lambda_r(A) \to 0, \tag{1}$$

$$\Lambda_r(A)^{(n)} \to A \to 0. \tag{2}$$

Both of these sequences are split: (1) because Λ_r is a projective Λ_r-module, and (2) by Proposition 2 of ([8]). Taking into account the uniqueness of decomposition into indecomposables, we obtain our assertion.

Proposition 3. *Let Λ satisfy condition $(*)$; let A, B be indecomposable finitely generated Λ-modules without o-torsion. Then, if $A \circ B \neq B$, then $A \circ B = A^{(s)}$.*

Suppose first that $B \circ A = A$; then from Proposition 2 it follows that

$$\Lambda_r(B) \circ \Lambda_r(A) = \Lambda_r(A),$$

and hence, by Lemma 1,

$$\Lambda_r^*(A) \circ \Lambda_r^*(B) == \Lambda_r^*(A).$$

Taking into account that

$$\Lambda_r^*(A) \simeq \Lambda_r(A) \simeq A^{(m)}, \qquad \Lambda_r^*(B) \simeq \Lambda_r(B) \simeq B^{(n)},$$

we have

$$A^{(m)} \circ B^{(n)} = (A \circ B)^{(n)} = A^{(m)}.$$

Taking into account the indecomposability of A and the uniqueness of decomposition into indecomposables, we obtain

$$A \circ B = A^{(s)},$$

where $s = m/n$.

Suppose now that $B \circ A \neq A$. Then the module $C = A \oplus B$ is normally indecomposable.

$$C \circ A = A, \qquad \Lambda_r(C) \circ \Lambda_r(A) = \Lambda_r(A), \qquad \Lambda_r^*(A) \circ \Lambda_r^*(C) = \Lambda_r^*(A),$$

$$A^{(m)} \circ \left(A^{(n_1)} \oplus B^{(n_2)}\right) = A^{(n_1)} \oplus (A \circ B)^{(n_2)} = A^{(m)}, \qquad A \circ B = A^{(s)}.$$

Let $a = (a_1, \dots, a_t)$, $b = (b_1, \dots, b_t)$ be vectors with nonnegative integral coordinates. We shall say that the vectors a, b are in the relation ρ, and write $a\rho b$, if there exists an integer n such that from $a_i \neq 0$ it follows that

$$b_i = a_i n.$$

Lemma 2. *Let $a^1, \dots, a^s, \lambda$ be t-dimensional vectors with nonnegative integral coordinates, where $a_i \rho \lambda$ for $1 \leq i \leq t$, and for any i, j either $a_i \rho a_j$ or $a_j \rho a_i$. Suppose further that nonnegative integers $m_1, \dots, m_s$ are given, the vector*

$$u = \sum_{i=1}^{s} m_i a^i$$

is constructed, and the vector v, where

$$v_j = \min(u_j, \lambda_j).$$

Then there exist integers $m_1', \dots, m_s'$, $m_i' \leq m_i$, such that

$$\sum_{i=1}^{s} m_i' a_i = v.$$

The lemma is proved without difficulty, for example, by induction on

$$\sum_{i=1}^{t} \lambda_i.$$

Proposition 4. *Suppose that for every prime ideal P of the ring o the corresponding o_p-order*

$$\Lambda_p = \Lambda \otimes_o o_p,$$

where o_p is the localization with completion of the ring o, satisfies condition $(*)$. Then every finitely generated Λ-module M without o-torsion decomposes into a direct sum of ideals of the order Λ.

Put

$$\widetilde{M} = M \otimes_o k, \qquad M_p = M \otimes_o o_p, \qquad \widetilde{M}_p = M_p \otimes_{o_p} k_p, \qquad \widetilde{\Lambda}_p = \Lambda_p \otimes_{o_p} k_p,$$

where k is the quotient field of o_p. Denote by $\tilde{I}(M)$ $(\tilde{I}(M_p))$ the largest ideal of the algebra $\widetilde{\Lambda}$ $(\widetilde{\Lambda}_p)$ splitting off as a direct summand from $\widetilde{M}$ $(\widetilde{M}_p)$. We shall show that

$$M = I \oplus X,$$

where

$$I \otimes_o k \simeq \tilde{I}(M).$$

Note that

$$\tilde{I}(M_p) \simeq \tilde{I}(M) \otimes_o k_p.$$

Therefore, if we show that for every P

$$M_p = I_p \oplus X_p,$$

where

$$I_p \otimes_{o_p} k_p \simeq \tilde{I}(M_p),$$

then, using the results ([9]), we shall be able to "glue" the global decomposition

$$M = I \oplus X, \qquad I \otimes k \simeq \tilde{I}(M).$$

Let $\widetilde{\Lambda}_p$ decompose into a direct sum of t simple algebras. To every Λ_p-module we assign a t-dimensional integer vector in which the i-th coordinate indicates the number of simple direct summands corresponding to the i-th simple algebra in the decomposition of $\widetilde{\Lambda}_p$. If we now denote by $A_1, \dots, A_s$ the indecomposable Λ_p-modules occurring in the decomposition of the module M_p into indecomposables, by a^i the vector corresponding to

$$A_i = A_i \otimes_{o_p} k_p,$$

and by λ the vector corresponding to $\widetilde{\Lambda}_p$, and take into account that from Proposition 2 one may conclude that $a^i \rho \lambda$, and from Proposition 3 that for any ...

i and j, or $a^i \rho a^j$, or $a^j \rho a^i$, then it is not hard to see that our assertion follows directly from Lemma 2.

The assertion of the theorem now follows from Propositions 1 and 4.

Remark 1. From the proof it is clear that it suffices to require the existence of two generators for the localizations over the prime ideals of the ring Λ.

Remark 2. The condition of the theorem is not necessary even in the commutative case. Thus, for example, for the integral group ring of the cyclic group of order four, not every ideal has two generators; nevertheless every representation module decomposes into a direct sum of ideals [10].

Remark 3. From Proposition 3 one can conclude that if, in the complete local case, an indecomposable order Λ satisfies the conditions of the theorem, then the number of simple components in the corresponding semisimple algebra cannot be greater than 2.

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Note: Figure translations are in progress. See original paper for figures.