ON ABSOLUTE AND UNCONDITIONAL CONVERGENCE

MATHEMATICS

1966

SovietRxiv

Abstract

Full Text

UDC 517.512.2

MATHEMATICS

P. L. ULYANOV

ON ABSOLUTE AND UNCONDITIONAL CONVERGENCE

(Presented by Academician P. S. Novikov on 6 XI 1965)

In this note we shall present several results concerning, mainly, absolute and unconditional convergence of series with respect to the Haar system $\{\chi_m(t)\}$ (see (1), pp. 54-58). The results obtained in this direction show, in particular, that for series with respect to the Haar system the following types of convergence are essentially different from one another: absolute convergence everywhere, absolute convergence of the series of coefficients, absolute convergence almost everywhere (for trigonometric series these types of convergence are equivalent to one another).

We consider the Haar system in the enumeration

$$\chi_1(t) \equiv \chi_0^{(0)}(t), \qquad \chi_m(t) \equiv \chi_n^{(k)}(t) \quad (t \in [0,1]),$$

where $m = 2^n + k$ with $1 \le k \le 2^n$ and $n = 0, 1, \ldots$.

Let us note that the assertions formulated below directly adjoin the author’ s results published in papers ([3]–[5]).

By $\omega_p(\delta, f)$ we shall denote the integral (in L^p) modulus of continuity of a function $f(t) \in L^p(0,1)$ with $p \in [1,\infty]$. For convenience we assume that $\omega_\infty(\delta, f) = \omega(\delta, f)$, the ordinary modulus of continuity of a function $f(t) \in C(0,1)$ in the uniform metric.

§ 1. Let $f(t) \in L^p(0,1)$ for some $p \in [1,\infty]$ and

$$f(t) \sim \sum_{m=1}^{\infty} a_m(f)\chi_m(t) \tag{1}$$

be its Fourier series with respect to the Haar system, i.e. $a_m(f) = (f, \chi_m)$ for $m \ge 1$.

In this paragraph we formulate conditions guaranteeing the almost-everywhere convergence of the series

$$\sum_{m=1}^{\infty} |a_m(f)\chi_m(t)|. \tag{2}$$

Namely, the following is true.

Theorem 1. *If for some* $p \in (1, \infty]$ *the series*

$$\sum_{m=2}^{\infty} \frac{1}{m\sqrt{\lg m}}\, \omega_p\left(\frac{1}{m}, f\right) < \infty, \tag{3}$$

then the series (2) converges almost everywhere on $[0, 1]$.

This assertion is, in a certain sense, definitive, since the following holds.

Theorem 2. *Let a positive function* $\omega(\delta)$ *on* $(0, 1]$ *be such that*

$$\omega(\delta)\left(\lg\frac{1}{\delta}\right)^{\alpha} \downarrow \quad \text{as } \delta \downarrow 0 \quad (0 < \delta < \delta_0) \tag{4}$$

for some $a > 0$ and

$$\lim_{\delta \to +0} \frac{\omega(2\delta)}{\omega(\delta)} = 1. \tag{5}$$

Then, if

$$\sum_{m=2}^{\infty} \frac{1}{m\sqrt{\lg m}}\, \omega\left(\frac{1}{m}\right) = \infty, \tag{6}$$

there exists a function $f(t) \in L^p(0, 1)$ for all $p \in [1, \infty)$ such that

$$\omega_p(\delta, f) = O\{\omega(\delta)\} \qquad \text{for all } p \in (1, \infty),$$

but the series (1) is not unconditionally convergent almost everywhere on $[0, 1]$. Moreover, after a certain rearrangement of its terms, the series (1) diverges almost everywhere on $[0, 1]$.

For $p = 2$, special cases of Theorems 1 and 2 were established by us in the paper [(5)].

§ 2. We now consider the question of convergence of the series (2) for all $t \in [0, 1]$. In this direction the following is valid.

Theorem 3. Let $f(t) \in L^p(0, 1)$ for some $p \in [1, \infty]$. Then, if

$$\sum_{m=1}^{\infty} m^{1/p-1}\omega_p\left(\frac{1}{m}, f\right) < \infty, \tag{7}$$

then the series (2) converges uniformly on $[0, 1]$.

For the case $p = \infty$, Theorem 3 was proved by Chiselskii and Musielak [(2)].

Furthermore, the following hold.

Theorem 4. Let $\omega(\delta)$ be the modulus of continuity of some function continuous on $[0, 1]$. Then, if

$$\sum_{m=1}^{\infty} \frac{1}{m}\,\omega\left(\frac{1}{m}\right) = \infty,$$

there exists a function $f(t) \in C(0, 1)$ such that

$$\omega(\delta, f) = O\{\omega(\delta)\},$$

but the series (2) diverges at some point $t_0 \in [0, 1]$.

This theorem shows that the assertion of Chiselskii and Musielak (Theorem 3 for $p = \infty$) is, in a certain sense, unimprovable.

Theorem 5. Let $p \in (1, \infty)$ and let $\omega(\delta)$ be such that

$$\delta^{-1/p}\omega(\delta) \downarrow 0 \quad \text{as} \quad \delta \downarrow 0 \ (0 < \delta \leq \delta_0),$$

$$\overline{\lim_{\delta \to +0}} \frac{\omega(2\delta)}{\omega(\delta)} < 2. \tag{8}$$

Then, if

$$\sum_{m=1}^{\infty} m^{1/p-1}\omega\left(\frac{1}{m}\right) = \infty,$$

there exists a function $f(t)$, continuous on $[0, 1]$, with

$$\omega_p(\delta, f) = O\{\omega(\delta)\},$$

for which the series (2) does not converge at some point $t_0 \in [0, 1]$.

This assertion shows that Theorem 3 for $p \in (1, \infty)$ is, in a certain sense, unimprovable.

Remark 1. With regard to inequality (7) one may assert the following:

a) if for the function f inequality (7) is valid for some $p_0 \in (1,\infty)$, then inequality (7) is also valid for all $p \in (p_0,\infty)$, but not necessarily for $p=\infty$ and $p \in [1,p_0)$;

b) if for the function f inequality (7) is valid for $p=\infty$, this still does not imply that it will be valid also for some $p \in [1,\infty)$;

c) if for the function f inequality (7) is valid for some $p \in (1,\infty)$, then the function f is equivalent to some continuous function, and this assertion is in a certain sense definitive.

§ 3. Adjoining the preceding results are (in form) assertions concerning series with respect to the Rademacher system (see ([1]), p. 59). Namely, the following is true.

Theorem 6. *Let*

$$f(t)=\sum_{m=1}^{\infty} b_m r_m(t)$$

for almost all $t \in [0,1]$. *Then, if for some* $p \in [1,\infty)$ *inequality (3) is valid, then*

$$\sum_{m=1}^{\infty} |b_m| < \infty. \tag{9}$$

For the case $p=1$ this assertion is a strengthening of the result of Ciesielski and Musielak ([2]), which states that the inequality

$$\sum_{m=1}^{\infty} \frac{1}{m}\,\omega_1\left(\frac{1}{m},f\right) < \infty$$

implies inequality (9). Thus, Theorem 6 ($p=1$) shows that the result of Ciesielski and Musielak is not definitive.

The definitive result is Theorem 6, since the following holds.

Theorem 7. *If* $\omega(\delta)$ *satisfies the requirements (4), (5), and (6), then there exists a function* $f(t)$ *with*

$$f(t)=\sum_{m=1}^{\infty} b_m r_m(t)$$

(for almost all $t \in [0,1]$*) such that* $\omega_p(\delta,f)=O\{\omega(\delta)\}$ *for all* $p \in [1,\infty)$, *while inequality (9) is not satisfied.*

Remark 2. Assertions of the same type (see Theorems 6 and 7) also hold for lacunary trigonometric series. In this case condition (5) may be replaced by the less restrictive condition (8).

Remark 3. In Theorems 2 and 5 the conditions on the function $\omega(\delta)$ can be considerably weakened.

Received
4 XI 1965

REFERENCES

[1] G. Alexits, *Problems of convergence of orthogonal series*, Moscow, 1963.
[2] L. Ciesielski, J. Musielak, *Colloq. math.*, **7**, No. 1, 61 (1959).
[3] P. L. Ul' yanov, *Uspekhi Mat. Nauk*, **16**, issue 3, 61 (1961).
[4] P. L. Ul' yanov, *Mat. sbornik*, **63**, No. 3, 356 (1964).
[5] P. L. Ul' yanov, *Vestn. Mosk. Univ., Ser. Math.*, No. 4, 35 (1965).

Note: Figure translations are in progress. See original paper for figures.