ON MULTIDIMENSIONAL HYPERBOLIC EQUATIONS OF ARBITRARY ORDER WITH DISCONTINUOUS COEFFICIENTS

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Abstract

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ON MULTIDIMENSIONAL HYPERBOLIC EQUATIONS OF ARBITRARY ORDER WITH DISCONTINUOUS COEFFICIENTS

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The mixed problem for hyperbolic equations with discontinuous coefficients, despite its theoretical and practical significance ([1]), has at present still been insufficiently studied. This problem has been studied most fully in the case of two independent variables. In the multidimensional case, results have been obtained that pertain either to separate classes of second-order equations ([2–5]), or to systems of second order of a special form ([2]).

In the present note we consider the mixed problem for one class of multidimensional hyperbolic equations of arbitrary order in the case when the coefficients and the free term of the equation, as well as the initial functions, have jump discontinuities on certain surfaces of a fairly general form. At the same time the boundary conditions and the matching conditions on the surfaces of discontinuity are prescribed in a general form.

Let G be a given domain (not necessarily bounded) with analytic boundary S_0 in the n-dimensional Euclidean space of points $x = (x_1, \dots, x_n)$. By S we denote the union of the graphs of analytic $(n-1)$-dimensional surfaces S_r $(1 \le r \le p;\ 0 \le p < \infty)$ lying inside G. If G is a bounded domain, then all S_r $(0 \le r \le p)$ are closed; if G is unbounded, then any of the surfaces $S_1, \dots, S_p$ may be either closed or extend to infinity. It is assumed that all sets $S_i \cap S_j$ $(i \ne j;\ 0 \le i, j \le p)$ are empty. No other restrictions are imposed on the nature of the arrangement of the surfaces $S_0, S_1, \dots, S_p$ with respect to one another. Then the set $\overline{G} \setminus S$ consists of $p+1$ connected components G_r $(1 \le r \le p+1)$, whose numbering depends on the nature of the arrangement of the surfaces S_k $(1 \le k \le p)$. We note only that the component G_1 adjoins the surface S_0; to the surface S_k $(1 \le k \le p)$ there adjoins on one side the component G_{k+1}, and on the other side one of the components $G_1, \dots, G_k$. In what follows we shall denote the number of such a component by the index j_k. The indices j_k for different $k = 1, 2, \dots, p$ may, in general, coincide.

In the cylinder $G_T = \{\overline{G} \setminus S\} \times \{0 \le t \le T\}$ consider the hyperbolic equation

$$\sum_{|k|\le m} a_{(k)}(x,t)\,\frac{\partial^{|k|}u}{\partial t^{k_0}\partial x_1^{k_1}\cdots\partial x_n^{k_n}} = f(x,t) \qquad (1)$$

$$(|k| = k_0 + k_1 + \dots + k_n; \quad (k) = (k_0, k_1, \dots, k_n); \quad a_{m,0,\dots,0}(x,t) \equiv 1).$$

The coefficients $a_{(k)}(x,t)$ and the free term $f(x,t)$ are assumed to be piecewise analytic functions in the sense that they are given on G_T and are analytic in all their arguments in each $\overline{G}_{rT} = \overline{G}_r \times$

$\times \{0 \le t \le T\}$ $(1 \le r \le p+1)$; they may have discontinuities of the first kind on the surfaces S_i $(1 \le i \le p)$, i.e.,

$$a_{(k)}(x,t) = a^r_{(k)}(x,t); \qquad f(x,t) = f^r(x,t) \quad \text{for } (x,t) \in \overline{G}^T_r$$

$$(1 \le r \le p+1).$$

Hyperbolicity of equation (1) is understood in the sense that in the decompositions

$$\sum_{|k|=m} a^r_{(k)}(x,t)\lambda^{k_0}\xi_1^{k_1}\cdots\xi_n^{k_n} = \prod_{j=1}^{m}[\lambda - \lambda^r_j(x,t;\xi)] \qquad (1 \le r \le p+1).$$

the functions $\lambda^r_j(x,t;\xi)$ $(1 \le j \le m)$ are real and distinct for all $(x,t) \in \overline{G}^T_r$ and for all $\xi_1, \dots, \xi_n$ from the unit sphere. We shall assume these functions renumbered so that

$$\lambda^r_1(x,t;\xi) < \dots < \lambda^r_m(x,t;\xi).$$

It is assumed that, for given j, r, x, and ξ, the function $\lambda^r_j(x,t;\xi)$, as a function of the single variable t, is sign-constant on the interval $[0,T]$.

Denote by $\nu^k(y)$ $(\nu^k_1(y), \dots, \nu^k_n(y))$ $(0 \le k \le p)$ the unit normal vector to the surface S_k at the point y, directed inward into the domain G_{k+1}. Then divide the surface S_0 into $m+1$ parts $S_{00}, S_{01}, \dots, S_{0m}$ according to the following rule: S_{0j} $(0 \le j \le m)$ contains all points $y \in S_0$ at which, among the functions $\lambda^1_1(y,t;\nu^0(y)), \dots, \lambda^1_m(y,t;\nu^0(y))$, exactly j are negative. Each of the surfaces S_k $(1 \le k \le p)$ is divided into $2m+1$ parts $S_{k0}, S_{k1}, \dots, S_{k,2m}$ by the rule: S_{kj} $(0 \le j \le 2m)$ contains all those points $y \in S_k$ at which the total number of positive quantities among $\lambda^k_1(y,t;\nu^k(y)), \dots, \lambda^k_m(y,t;\nu^k(y))$ and negative quantities among $\lambda^{k+1}_1(y,t;\nu^k(y)), \dots, \lambda^{k+1}_m(y,t;\nu^k(y))$ is equal to j. Clearly, some of the sets S_{0j} $(0 \le j \le m)$ and S_{kj} $(1 \le k \le p;\ 0 \le j \le 2m)$ may be empty; on the other hand, the introduced sets need not be connected.

Let $S^T_{rj} = S_{rj} \times \{0 \le t \le T\}$.

For equation (1) we impose initial conditions

$$\left.\frac{\partial^i u^r}{\partial t^i}\right|_{t=0} = g^r_i(x) \qquad (x \in \overline{G}_r;\ 1 \le r \le p+1;\ 0 \le i \le m-1), \qquad (2)$$

boundary conditions

$$\sum_{|k|\le m-1} b^j_{(k)}(y,t)\,\frac{\partial^{|k|}u^1(y,t)}{\partial t^{k_0}\partial y_1^{k_1}\cdots\partial y_n^{k_n}} = h^0_j(y,t) \tag{3}$$

$$\left((y,t)\in\bigcup_{r=j}^{m} S^T_{0r};\ 1\le j\le m\right)$$

and conjugation conditions on the discontinuity surfaces

$$\sum_{|s|\le m-1} c^{k,j}_{(s)}(y,t)\,\frac{\partial^{|s|}u^k(y,t)}{\partial t^{s_0}\partial y_1^{s_1}\cdots\partial y_n^{s_n}} = \sum_{|s|\le m-1} d^{k,j}_{(s)}(y,t)\,\frac{\partial^{|s|}u^{k+1}(y,t)}{\partial t^{s_0}\partial y_1^{s_1}\cdots\partial y_n^{s_n}} + h^k_j(y,t) \tag{4}$$

$$\left((y,t)\in\bigcup_{r=j}^{2m} S^T_{kr};\ 1\le j\le 2m;\ 1\le k\le p\right).$$

On S^T_{00} boundary conditions are not imposed; correspondingly, on S^T_{k0} $(1\le k\le p)$ conjugation conditions are not imposed.

If some of the boundary conditions (or conjugation conditions) contain only derivatives of the unknown function of order lower than $m-1$, then, by differentiating with respect to t, such conditions can always be brought to the form (3) (or (4)).

Each of the functions $g^r_j(x)$ $(1\le r\le p+1;\ 0\le j\le m-1)$; $b^j_{(k)}(y,t)$ $(|k|\le m-1;\ 1\le j\le m)$; $h^0_j(y,t)$ $(1\le j\le m)$; $c^{k,j}_{(s)}(y,t)$, $d^{k,j}_{(s)}(y,t)$ $(|s|\le m-1;\ 1\le k\le p;\ 1\le j\le 2m)$; $h^k_j(y,t)$ $(1\le j\le 2m;$

$1\le k\le p)$ is assumed to be analytic in its domain of definition with respect to all its arguments. In addition, it is assumed that the initial conditions (2) are naturally compatible with the boundary conditions (3) and the conjugation conditions (4) at the points of the surfaces S_r $(0\le r\le p)$:

$$\sum_{|k|\le m-1} b^j_{(k)}(y,0)\,\frac{\partial^{k_1+\cdots+k_n}g^1_{k_0}(y)}{\partial y_1^{k_1}\cdots\partial y_n^{k_n}} = h^0_j(y,0) \quad \left(y\in\bigcup_{r=j}^{m} S_{0r};\ 1\le j\le m\right),$$

$$\sum_{|s|\le m-1} c^{k,j}_{(s)}(y,0)\,\frac{\partial^{s_1+\cdots+s_n}g^{jk}_{s_0}(y)}{\partial y_1^{s_1}\cdots\partial y_n^{s_n}} = \sum_{|s|\le m-1} d^{k,j}_{(s)}(y,0)\,\frac{\partial^{s_1+\cdots+s_n}g^{k+1}_{s_0}(y)}{\partial y_1^{s_1}\cdots\partial y_n^{s_n}} + h^k_j(y,0) \tag{5}$$

$$\left(y\in\bigcup_{r=j}^{2m} S_{kr};\ 1\le j\le 2m;\ 1\le k\le p\right).$$

Let the numbers α_{kj} and β_{kj} denote, respectively, the number of positive among the quantities $\lambda_r^{jk}(y,t;\nu^k(y))$ and negative among the quantities $\lambda_r^{k+1}(y,t;\nu^k(y))$ for $(y,t) \in S_{kj}^T$ $(1 \le r \le m;\ 1 \le k \le p;\ 1 \le j \le 2m;\ 0 \le \alpha_{kj}, \beta_{kj} \le m)$. By definition, $\alpha_{kj} + \beta_{kj} = j$.

Introduce the notation:

$$B_{sq}^{0j}(y,t) = \sum_{r=0}^{m-1} \sum_{k_1+\dots+k_n=r} b^s_{m-r-1,k_1,\dots,k_n}(y,t)[\nu_1^0(y)]^{k_1} \cdots [\nu_n^0(y)]^{k_n} \times [\lambda_q^1(y,t;\nu^0(y))]^{m-r-1}$$

$$((y,t) \in S_{0j}^T;\ 1 \le j \le m;\ 1 \le s \le j;\ 1 \le q \le m);$$

$$C_{sq}^{kj}(y,t) = \sum_{r=0}^{m-1} \sum_{r_1+\dots+r_n=r} c^{k,s}_{m-r-1,r_1,\dots,r_n}(y,t)[\nu_1^k(y)]^{r_1} \cdots [\nu_n^k(y)]^{r_n} \times [\lambda_q^{jk}(y,t;\nu^k(y))]^{m-r-1},$$

$$D_{sq}^{kj}(y,t) = \sum_{r=0}^{m-1} \sum_{r_1+\dots+r_n=r} d^{k,s}_{m-r-1,r_1,\dots,r_n}(y,t)[\nu_1^k(y)]^{r_1} \cdots [\nu_n^k(y)]^{r_n} \times [\lambda_q^{k+1}(y,t;\nu^k(y))]^{m-r-1}$$

$$((y,t) \in S_{kj}^T;\ 1 \le k \le p;\ 1 \le j \le 2m;\ 1 \le s \le 2m;\ 1 \le q \le m).$$

The following conditions are assumed to be satisfied:

A. At every point $(y,t) \in S_{0j}^T$

$$\det \left\| B_{sq}^{0j}(y,t) \right\|_{s,q=1}^{j} \ne 0 \quad (1 \le j \le m),$$

B. At every point $(y,t) \in S_{kj}^T$

$$\begin{vmatrix} C^{kj}_{1,m-\alpha_{kj}-1}(y,t) & \cdots & C^{kj}_{1,m}(y,t) & D^{kj}_{11}(y,t) & \cdots & D^{kj}_{1,\beta_{kj}}(y,t) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ C^{kj}_{j,m-\alpha_{kj}-1}(y,t) & \cdots & C^{kj}_{j,m}(y,t) & D^{kj}_{j1}(y,t) & \cdots & D^{kj}_{j,\beta_{kj}}(y,t) \end{vmatrix} \ne 0,$$

$$(1 \le k \le p;\ 1 \le j \le 2m).$$

Theorem. Let the coefficients and free terms of equation (1), the boundary conditions (3) and the conjugation conditions (4), as well as the initial functions (2) and the surfaces S_k $(0 \le k \le p)$, satisfy the smoothness conditions formulated above; let conditions A, B and the compatibility conditions (5) be satisfied.

Then problem (1)–(4) has a solution $u(x,t)$, analytic in each G_k^T $(1 \le k \le p+1)$, except for the characteristic-

those issuing from the points $S_0, S_1, \dots, S_p$ and falling inside $\bigcup_{r=1}^{p+1} G_r^\tau$. On the indicated characteristics the solution is continuous.

The method used to prove the theorem is an analogue of the well-known method of half-space potentials, applied in (6) to elliptic boundary-value problems. We note only that in the hyperbolic case this method leads to Volterra integro-differential equations, which, in the case of analytic data, are solved by the method of successive approximations.

Finally, let us observe that conditions A, B are essential for the solvability of the problem. One can give an example of a problem in which at least one of these conditions is not satisfied and which is therefore unsolvable.

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Note: Figure translations are in progress. See original paper for figures.