ON THE REGULARIZATION OF THE EARTH

GEOPHYSICS

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Fig. 1

Figure 1: Fig. 1

Abstract

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GEOPHYSICS

M. A. ALEKSIDZE

ON THE REGULARIZATION OF THE EARTH

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The basic requirements imposed on methods for the regularization of the Earth are the following ([1]):

1. Outside the surface of the geoid there must be no attracting masses.

2. The mass of the Earth must be constant.

3. The surface of the geoid must not change.

4. The geoid and the spheroid must have a common center of mass.

None of the developed methods of regularization satisfies all the requirements listed. Thus, for example, Rudzkij' s inversion method ([1,2]) satisfies the first and third conditions. A shortcoming of this method is considered to be ([1]) that the second condition is not fulfilled in it.

Fig. 1

In the present note it is first shown that, in principle, it is impossible to develop such a method of regularization that would satisfy the first three requirements listed above. Taking this circumstance into account, it is natural to strive to satisfy the above requirements approximately. Such modifications of the inversion method are given which strictly satisfy the first two conditions, while the third condition is satisfied, in a certain sense, in the best possible way.

Let S be an arbitrary closed surface, G the finite three-dimensional region bounded by it, and

$$U(S) = k \iiint_{G_1} \frac{dm}{r}$$

the potential on S of some mass distribution in the finite region G_1. We shall assume that G and G_1 have no common points. It is required to find such a new mass distribution that the following conditions are fulfilled:

$$1. \quad V(S) = k \iiint_{G_2} \frac{dm}{r} = U(S).$$

$$2. \quad \iiint_{G_1} dm = \iiint_{G_2} dm = M.$$

$$3. \quad G_2 \equiv G.$$

Suppose such a distribution has been constructed. Applying the sweeping-out operation ([3,4]), one can find such a mass distribution m_2 on the surface S that

$$\iint_S \frac{dm}{r} = \iiint_{G_2} \frac{dm}{r}, \qquad \iint_S dm_2 = \iiint_{G_2} dm_1 = M.$$

It is known ([3]) that the first of the last conditions uniquely determines the distribution m_2, while the second condition is satisfied automatically. Thus we have obtained such a surface distribution of masses that the corresponding simple-layer potential on S coincides with the Newtonian po-

distribution potential m and

$$\iint\limits_S dm_2 = \iiint\limits_{G_1} d\tau.$$

Consequently, we have performed the operation of sweeping outward while preserving the total mass. The latter is impossible, for if under internal sweeping the total mass remains constant, then under outward sweeping the total mass decreases ([3]).

Let it be required to transfer, from the point A (Fig. 1) to the internal point B, a mass m in such a way that the difference of the potentials of the points A and B on the circle S be minimal in the sense of the metric C, i.e., it is necessary to find such a $\rho_1 < R$ that would give a minimum to the expression $\max_\psi |\omega(\rho_1, \psi)|$, where

$$\omega(\rho_1, \psi) = \frac{1}{\sqrt{\rho_1^2 + R^2 - 2\rho_1 R \cos\psi}} - \frac{1}{\sqrt{\rho^2 + R^2 - 2\rho R \cos\psi}},$$

$$R = OD, \quad \rho_1 = OB, \quad \rho = OA \quad \text{(Fig. 1)}$$

The stationary values for ψ are found from the equation

$$\frac{\partial\omega(\rho_1,\psi)}{\partial\psi} = \frac{\sin\psi\,[\rho_1 R(\rho^2+R^2-2\rho R\cos\psi)^{3/2} - \rho R(\rho_1^2+R^2-2\rho_1 R\cos\psi)^{3/2}]}{(\rho_1^2+R^2-2\rho_1 R\cos\psi)^{3/2}(\rho^2+R^2-2\rho R\cos\psi)^{3/2}} = 0. \tag{1}$$

Since $\rho_1 < R < \rho$, in order that (1) be satisfied either

$$\sin\psi = 0, \qquad \psi_1 = 0, \qquad \psi_2 = \pi,$$

or

$$\cos\psi = \frac{1}{2R}\frac{\rho_1^{2/3}(\rho^2+R^2) - \rho^{2/3}(\rho_1^2+R^2)}{\rho\rho_1^{2/3} - \rho_1\rho^{2/3}},$$

$$\psi_{3,4} = \pm\arccos\frac{1}{2R}\frac{\rho_1^{2/3}(\rho^2+R^2) - \rho^{2/3}(\rho_1^2+R^2)}{\rho\rho_1^{2/3} - \rho_1\rho^{2/3}}. \tag{2}$$

It is not difficult to verify the equalities

$$\cos\psi = 1 + O(\varepsilon+\varepsilon_1), \qquad \cos\psi = 1 + O[(\varepsilon+\varepsilon_1)^2],$$

$$\cos\psi = \frac{6 - 11(\varepsilon_1-\varepsilon)}{6 - 2(\varepsilon_1-\varepsilon)} + O[(\varepsilon+\varepsilon_1)^3], \tag{3}$$

$$\cos\psi = \frac{1/3 - 11/8(\varepsilon_1-\varepsilon) - 4/81(\varepsilon_1^2+\varepsilon^2) - 32/81\,\varepsilon\varepsilon_1}{1/3 - 1/9(\varepsilon_1-\varepsilon) - 4/81(\varepsilon_1^2+\varepsilon^2) - 5/81\,\varepsilon\varepsilon_1} + O[(\varepsilon+\varepsilon_1)^3],$$

where $\varepsilon_1 = H_1/R,\ \varepsilon = H/R,\ H = AC,\ H_1 = BC$ (Fig. 1).

For the stationary values ψ_1 and ψ_2, for $\omega(\rho_1,\psi)$ we obtain

$$\omega_1(\rho_1,\psi_1) = 1/H_1 - 1/H, \qquad \omega_2(\rho_1,\psi_2) = 1/(2R-H_1) - 1/(2R+H).$$

For $H = H_1$,

$$\omega_1 = 0, \qquad \omega_2 = \frac{2H}{4R^2-H^2}.$$

Since in this case $\omega(\psi) \geq 0$, it is clear that ψ_1 and ψ_2 correspond to minimal values of ω, and ψ_3 and ψ_4 to maximal values. From (2) it is seen that, for $H_1 = H \to 0$ ($\rho \to R,\ \rho_1 \to R$), $\psi_3 \to 0$. For example, for $H_1 = H = 10$ km, $R = 6372$ km,

$$\psi_{3,4} = \pm \arccos 0.9999856,$$

and the maximum value for ω will be

$$\omega_3 = \frac{1}{\sqrt{\rho_1^2 + R^2 - 2\rho_1 R \cos\psi_3}} - \frac{1}{\sqrt{\rho^2 + R^2 - 2\rho R \cos\psi}} =$$

$$= \frac{1}{\sqrt{H_1^2 + 2\rho_1 R(1 - \cos\psi_3)}} - \frac{1}{\sqrt{H^2 + 2\rho R(1 - \cos\psi_3)}} = 0.4 \cdot 10^{-4}\ (\text{km})^{-1}.$$

The best value in the sense of the metric C for ρ is obtained from the equality $\omega_1 = -\omega_3$, or

$$\frac{1}{H_3} - \frac{1}{H} = \frac{1}{\sqrt{\rho^2 + R^2 - 2R\rho\cos\psi_3}} - \frac{1}{\sqrt{\rho_1^2 + R^2 - 2R\rho_1\cos\psi_3}}, \tag{4}$$

where ψ_3 is determined from (2). From equalities (3) it follows directly that the value $H_1 = H$ satisfies equation (4) with accuracy $O(\varepsilon + \varepsilon_1)$, i.e., for $H_1 = H$, (4) takes the form

$$O = O(\varepsilon + \varepsilon_1).$$

Depending on the problem being solved, it is sometimes more expedient, instead of the metric of the space C, to use the metric of the space L_2, i.e., to find such a ρ_1 that would give the minimum to the expression

$$W = \int_0^{2\pi} \left[\frac{1}{\sqrt{\rho^2 + R^2 - 2\rho R\cos\psi}} - \frac{1}{\sqrt{\rho_1^2 + R^2 - 2\rho_1 R\cos\psi}} \right]^2 d\psi =$$

$$= \frac{2\pi}{\rho^2 - R^2} + \frac{2\pi}{R^2 - \rho_1^2} - 4\int_0^{\pi} \frac{d\psi}{\sqrt{(\rho^2 + R^2 - 2\rho R\cos\psi)(\rho_1^2 + R^2 - 2\rho_1 R\cos\psi)}}. \tag{5}$$

To find the stationary values of ρ_1 in this case, we obtain the equation

$$\frac{2\pi\rho_1}{(R^2-\rho_1^2)^3} - \int_0^\pi \frac{\rho_1 - R\cos\psi}{\sqrt{\rho^2+R^2-2R\rho\cos\psi}\,(\rho_1^2-R^2-2R\rho_1\cos\psi)^{3/2}} = 0.$$

It is necessary to find the roots of this equation in the interval $-R < \rho_1 < R$ and then determine which of them gives the absolute minimum to expression (5).

Let us consider a physical model. At the point A ($H = 3$ km) there is a mass $m = 294 \cdot 10^{15}$ g (the mass of a homogeneous sphere of radius 3 km and density 2.6 g/cm^3). In the regularization of the Earth by Rudskii' s method,

$$H_1 = H(1 - H/R) = 2^{707}/_{708} \quad (R = 6371 \text{ km}),$$

and the mass at the point B will be

$$m_1 \approx m(1 - H/R) \approx 293.86 \cdot 10^{15} \text{ g}.$$

The change Δm in the total mass of the Earth is equal to $14 \cdot 10^{13}$ g. The mass $\Delta m = 14 \cdot 10^{13}$ g, located at the point A, gives at the point C a potential equal (in the CGS system) to

$$V = km/r = 32 \quad (k = 6.67 \cdot 10^{-8}).$$

If the Earth is regularized in the following way: $H = H_1 = 3$ km, $m_1 = m = 294 \cdot 10^{15}$, then at the point C the difference of potentials will be zero, and the maximum of the difference in the CGS system will be $km\omega_3 < 0.1$.

One of the main shortcomings of the inversion method is considered to be [1] the displacement of level surfaces adjacent to the geoid and the resulting change in the value of the vector of the acceleration of gravity. The problem may be posed as follows: move the mass m from the point A to an interior point B of the region G in such a way that the numerical value, corresponding to this mass, of the vector of the acceleration of gravity

$$km/\rho^2 + R^2 - 2R\rho\cos\psi$$

on the sphere S under consideration does not change. It is not hard to see that for this it is necessary to take $m_1 = (R^2/\rho^2)M$ and $\rho_1 = R^2/\rho$. The value ρ_1 coincided with the value of ρ_1 in Rudskii' s method, while the mass changed by almost twice the amount

$$\Delta m = m - m_1 = m\left(1 - \frac{R^2}{\rho^2}\right) \approx 2m\frac{H}{R}.$$

If the regularization of the Earth is carried out in such a way that the mass m does not change, and the numerical value of the difference $|g_A| - |g_B|$ ($|g|$ is the length of the vector g) is minimal in the sense of the metric C, then, to determine the stationary value ρ_1, we obtain the equation

$$\frac{1}{H^2} - \frac{1}{H_1^2} = \frac{1}{\rho^2 + R^2 - 2\rho R \cos\psi_4} - \frac{1}{\rho_1^2 + R^2 - 2\rho_1 R \cos\psi_4}, \tag{6}$$

where

$$\cos\psi_4 = \left[\sqrt{\rho}\,(\rho_1^2 + R^2) - \sqrt{\rho_1}\,(\rho^2 + R^2)\right]/2R(\sqrt{\rho_1}\rho - \sqrt{\rho}\rho_1).$$

It is easy to verify the equalities

$$\cos\psi_4 = 1 + O(\varepsilon + \varepsilon_1),$$

$$\cos\psi_4 = 1 + O\left[(\varepsilon + \varepsilon_1)^2\right],$$

$$\cos\psi_4 = \frac{8 - 5(\varepsilon + \varepsilon_1)}{8 - 2(\varepsilon + \varepsilon_1)} + O\left[(\varepsilon + \varepsilon_1)^3\right],$$

$$\cos\psi_4 = \frac{8 - 5(\varepsilon + \varepsilon_1) - (\varepsilon_1^2 + \varepsilon^2) - 4\varepsilon_1\varepsilon}{8 - 2(\varepsilon + \varepsilon_1) - (\varepsilon_1^2 + \varepsilon^2) - 3\varepsilon_1\varepsilon} + O\left[(\varepsilon + \varepsilon_1)^4\right].$$

From the last equalities it follows that the value $\rho_1 = R - H$ is close to the best one and satisfies (6) with accuracy $O(\varepsilon - \varepsilon_1)$. It should be noted, however, that in the last case we did not take into account the directions of the vectors $\mathbf{g}_A$ and $\mathbf{g}_B$. If the Earth is regularized under the condition of minimality (in the sense of the metric C) of the vector difference $\mathbf{g}_A - \mathbf{g}_B$, then we obtain $m_1 = 0$. Indeed, for any mass m_1 and any point $E \in G$ this difference at the point C is equal to

$$|\mathbf{g}_A - \mathbf{g}_B| = \sqrt{|\mathbf{g}_A|^2 + |\mathbf{g}_B|^2 - 2|\mathbf{g}_A|\,|\mathbf{g}_B|\cos\varphi}, \tag{7}$$

where φ is the angle between the vectors $\mathbf{g}_A$ and $\mathbf{g}_B$. Draw through the point C a tangent plane and denote by $\bar{G}$ the half-space not containing the point A. For any point $B \in \bar{G}$, the angle φ satisfies the condition $\pi/2 < \varphi < 3\pi/2$, and therefore $\cos\varphi < 0$. Since $G \in \bar{G}$, it is clear that the minimum (7) is attained when $m_1 = |\mathbf{g}_B| = 0$. Suppose that we carry out the regularization without changing the total mass and minimize, with respect to E, the vector difference $\mathbf{g}_A - \mathbf{g}_E$, where $E \in G$; it can be shown that in this case the mass m must be transferred from the point A to the point P (Fig. 1), which is determined from the condition

$$|\mathbf{g}_A(C)| + |\mathbf{g}_B(C)| = |\mathbf{g}_P(Q)| - |\mathbf{g}_A(Q)|$$

or

$$\frac{1}{H^2} + \frac{1}{(R^2 + H_2)^2} = \frac{1}{H_2^2} - \frac{1}{(R + H)^2},$$

where $H_2 = PQ$.

Computing Center
Academy of Sciences of the Georgian SSR

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Note: Figure translations are in progress. See original paper for figures.