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Abstract

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MATHEMATICS

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ON A CLASS OF DISCRETE GROUPS OF LOBACHEVSKY SPACE HAVING AN INFINITE FUNDAMENTAL DOMAIN OF FINITE MEASURE

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In ⁽¹⁾ one method was indicated for the geometric construction of an infinite set of nonisomorphic discrete groups of Lobachevsky space having a finite fundamental domain (Fedorov groups). In the present note we construct infinitely many discrete nonisomorphic groups of Lobachevsky space, the fundamental domain of each of which, although noncompact, has finite volume.

Let us proceed to the construction of the indicated group. Consider a sufficiently fine subdivision of an orisphere into squares (in the sense of the intrinsic metric of the orisphere, which, as is known, is Euclidean), and draw tangent planes at the centers of the squares, extending them to their mutual intersections. In this way we obtain a regular polyhedral simply connected surface with square faces (in the sense of Lobachevsky space; ⁽²⁾, p. 393), circumscribed about the orisphere, which bounds an infinite convex regular polyhedron (⁽²⁾, p. 440). If the sizes of the orispherical squares are increased, then at a certain moment all vertices of the polyhedron become infinitely distant. Let φ_0 be the value of the dihedral angle between adjacent faces at this moment. With a further increase in the sizes of the orispherical squares, the vertices become ideal, while the dihedral angles, continuing to decrease, can be made arbitrarily small. Therefore one can choose the size of the orispherical squares so that the dihedral angle is equal to $2\pi/k$, where k is any integer greater than $2\pi/\varphi_0$. In what follows we shall regard the number k as fixed. The edges that previously belonged to one vertex will now be orthogonal to one plane intersecting the polyhedron in a square. Consider only the part of this polyhedron truncated by such planes. We obtain a convex isotoxal polyhedron, semi-inscribed about the orisphere, whose faces are squares and equiangular semiregular (all angles are equal, and the sides are equal alternately) octagons. We shall call such a polyhedron the polyhedron M_k . With the polyhedron M_k one can associate three coaxial orispheres: one (the original one) is tangent to all octagonal faces at their centers, another is tangent to the square faces at their centers, and on the third lie all vertices of the polyhedron

(the polyhedron is inscribed in it). If one now draws through the vertices of the polyhedron M_k the axes of these orispheres and the corresponding planes, then on each of the orispheres there is induced a subdivision into orispherical squares and semiregular octagons, and each face lies inside its projecting pyramid. But since these pyramids do not enter one another, two nonadjacent faces cannot intersect either.

The resulting polyhedron M_k naturally decomposes into simpler infinite polyhedra P_k , corresponding to the squares of the original subdivision of the orisphere. Each such polyhedron P_k has nine faces, one infinitely distant vertex, and is cut out of the polyhedron M_k by planes each of which is determined by one

by the side of the corresponding horospherical square and by the axis of the horosphere passing through the endpoint of the side.

We now perform reflections in all square faces of the polyhedron M_k . Since the edges of the polyhedron are orthogonal to the planes of the square faces, after each successive reflection one obtains a convex polyhedron without self-intersections, and in the final result one obtains a convex generalized regular, in the sense of (3), polyhedron O_k , without vertices, whose edges are straight lines and all whose dihedral angles are equal to $2\pi/k$. We shall now attach to the polyhedron O_k , along whole faces, polyhedra equal to it (one may simply carry out reflections in faces or rotations about edges through angles equal to $2\pi/k$), and with the newly attached polyhedra O_k we shall do the same, continuing this process without bound. As a result we obtain a certain system of polyhedra O_k in Lobachevskii space. Since each polyhedron O_k is regularly composed of polyhedra M_k , and the latter in turn are regularly composed of polyhedra P_k , and since the algorithm for constructing the system of polyhedra O_k is the same no matter from which polyhedron O_k we start, the systems of polyhedra P_k and M_k obtained in this way are regular.

Let us note one important property of the obtained system of polyhedra M_k . We shall show that there exists a fixed number $a > 0$ such that, wherever the center of a sphere of radius a is placed in a polyhedron M_k , it will be covered by that polyhedron and by the polyhedra adjacent to it at vertices, edges, or faces. For this purpose we shall show that the polyhedra under consideration cover the $2a$ -neighborhood of any face of the polyhedron M_k (the a -neighborhood of any interior point of the polyhedron M_k at distance greater than a from its boundary will be covered by the polyhedron itself). For the proof we consider the given face and all faces of the given polyhedron adjacent to it, the union of all projecting pyramids corresponding to these faces, and the open set obtained after removing the boundary of the resulting union. Since the faces adjacent to the given one form on the surface of the polyhedron an impenetrable layer around the face under consideration, the latter lies entirely in the open set under consideration. Then the complement of this set to the whole space, being a closed set, lies at a finite distance $\mu > 0$ from the face under consideration, since the face is a closed finite set. Then, if we take $a = \mu/3$, the $2a$ -neighborhood of the face will lie entirely inside the open body under consideration and will be covered by the

part of this body lying inside the polyhedron, and by parts of analogous bodies belonging to polyhedra having with the given face at least one common point (for along edges and vertices, according to the construction, the space is divided by the faces of the polyhedra meeting there), which is what was required to prove.

After the preceding remark it is not difficult to show, following the work (4) of A. D. Aleksandrov, that the constructed system of polyhedra is a covering, and not only a covering but also a tiling of Lobachevskii space.

Indeed, take an arbitrary point A of the space and a point B already covered by some polyhedron. Draw the segment AB and divide it by points A_1, A_2, \dots, A_n into segments of length less than a . Since the point B is covered, the sphere of radius a with center at the point B is also covered by the polyhedra of the system. Consequently, the point A_1 is covered. Now we verify in the same way that the point A_2 is covered by the polyhedra of the system, and so on. Having reached the point A , we verify that it too is covered by the polyhedra of the system; and since the point A is arbitrary, it is thereby proved that the constructed system of polyhedra is a covering of the space.

Moreover, this covering of the space is a local decomposition of the a -neighborhood of every face, every edge, and every vertex of every polyhedron P_k . Indeed, at any vertex of a polyhedron P_k coinciding with a vertex of a polyhedron M_k , the polyhedra P_k do not enter into one another, since they meet as the central tetrahedral angles of a $2k$ -gonal bipyramid. At all the other vertices they do not enter into one another, since they meet as the central tetrahedral angles of an octahedron. Considering an edge, we see that if this edge is a ray, then it is covered by the corresponding polyhedron M_k , and to the star of the polyhedra P_k meeting in it there corresponds the star of squares of the original decomposition of the horosphere, and the polyhedra P_k do not enter into one another, since the squares do not enter into one another. If we consider an edge of the polyhedron P_k belonging to a square face of the polyhedron M_k , then, since the construction was carried out by reflections in square faces, at such an edge one obtains a star consisting of exactly four polyhedra P_k . If, however, one considers the star at an edge of the polyhedron P_k which is part of an edge of the polyhedron O_k , then such a star consists of exactly $2k$ polyhedra—in view of the choice of the value $2\pi/k$ for the dihedral angle of the polyhedron O_k . And since at the ends of edges, i.e. at the vertices of the polyhedron, there is a local decomposition of the space, it also occurs in a neighborhood of every edge. Since, moreover, the covering is normal, i.e. to each face of a given polyhedron there is attached only one neighboring polyhedron, covering the face under consideration as a whole, the covering will be a local decomposition also near every face of the polyhedron. Since the polyhedra M_k and O_k are decomposed into the polyhedra P_k , the covering by them will likewise be a local decomposition.

If we now construct, in the same way, from the polyhedra P_k an abstract complex

\tilde{K}^3 , then the map of it into Lobachevskii space, identifying the corresponding polyhedra, will be, as has been shown, a local homeomorphism. But since we have a map of the polyhedron \tilde{K}^3 onto the whole of Lobachevskii space, which is a complete connected space, it follows, by the well-known theorem of topology stating that the only covering of a simply connected manifold is the polyhedron itself, that the indicated map is one-to-one; and the covering constructed is a decomposition of Lobachevskii space.

But since the polyhedron P_k has finite volume, the fundamental domain of the discrete group of the constructed decomposition (being the eighth part of the polyhedron P_k) also has finite volume.

Let us now consider some edge of the polyhedron M_k to which two of its octagonal faces are adjacent, and draw the plane orthogonal to this edge at its midpoint. This plane intersects each of the polyhedra M_k meeting at the edge under consideration in a regular oricyclic polygon. At the vertex corresponding to the edge under consideration, exactly k such oricyclic polygons meet, which pass into one another under the motions carrying the corresponding polyhedra into one another. If we now pass along the sides of such polygons from the vertex under consideration to neighboring ones, the same picture will appear in them. We shall obtain a normal and regular decomposition of the plane into regular oricyclic polygons. Each of them naturally decomposes into singly asymptotic right triangles with acute angle equal to π/k , i.e. into precisely those figures which are fundamental domains of Hecke groups. Since in this case any such asymptotic triangle can be carried into any other asymptotic triangle belonging to the same oricyclic polygon by a motion carrying the oricyclic polygon into itself, the decomposition of the plane under consideration is regular with respect to the asymptotic

triangles. Thus, the spatial group constructed contains the Hecke group as its planar subgroup.

From all that has been said it is clear that, for any k , by our method one can construct a group containing an element of order k . At the same time, all arithmetic groups acting in three-dimensional Lobachevsky space and commensurable with the group of integral matrices with entries from an imaginary quadratic field can easily be checked not to contain elements of arbitrary orders. Therefore, among the groups constructed there are certainly nonarithmetic ones.

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