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Abstract

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MATHEMATICS

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ON SINGULAR POINTS OF TOPOLOGICAL EMBEDDINGS OF MANIFOLDS AND ON THE UNION OF LOCALLY FLAT CELLS

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1. This note continues ([1−3]). In it two problems are solved: on singular points of embeddings of manifolds and on the local flatness of the union of locally flat cells. The proofs here are only indicated. We use the terms and notation introduced in ([1]), and the technique of the proof of Theorem 1 from ([1]) (for a complete exposition see ([2])).

If $q : M^k \to E^n$ is an embedding of a k-dimensional manifold in Euclidean space, then a point $x \in M^k$ is called singular if it has in M a neighborhood V such that q is locally flat at all points of $V \setminus x$, except, possibly, x. If $k = n - 2$, $x \in \operatorname{Int} M$, then we shall say that $E^n \setminus qM$ has at the point qx the homotopy type of a circle if, for every sufficiently small neighborhood U of the point qx, there is a smaller neighborhood U' in E^n such that the image of $\pi_1(U' \setminus qM)$ in $\pi_1(U \setminus qM)$ under the homomorphism of inclusion is an infinite cyclic group, and the images of $\pi_k(U' \setminus qM)$ in $\pi_k(U \setminus qM)$, $k > 1$, are trivial.

2. **Main results.**

Theorem 1. *Let $q : M^k \to E^n$ be an embedding of a k-dimensional manifold, possibly with boundary, in n-dimensional Euclidean space.*

a) *If $n \geq 5$, $k \neq n - 2$, then the embedding q is locally flat at every singular point in the interior of the manifold.*

b) *If $n \geq 4$, then the embedding q is locally flat at every singular point on the boundary of the manifold.*

c) *If $n \geq 5$, $k = n-2$, $x \in \operatorname{Int} M$ is a singular point, then the embedding q is locally flat at x if and only if $E^n \setminus qM$ has at the point qx the homotopy type of a circle.*

Remark 1. For $k = n - 2$, singular points at which the embedding is not locally flat can indeed exist, for example, for a cone over a sphere knotted in an $(n-1)$-dimensional hyperplane (see also ([4])). If $n = 3$, then singular points are possible for $k = 1$ and 2 in the interior and for $k = 1, 2, 3$ on the boundary ([5]). The criterion given in part c), after an appropriate modification, can be transferred to the three-dimensional case. For the case $n \geq 4$, $k = 1$, see ([6]).

Remark 2. Some results on singular points were obtained earlier ([6–11]).

As Cantrell showed in ([7]), the proof of Theorem 1 reduces to the following theorem.

Theorem 2. *Let Q_1 and Q_2 be two locally flat k-dimensional cells in E^n, and suppose*

$$Q_1 \cap Q_2 = \dot{Q}_1 \cap \dot{Q}_2 = Q^{k-1}$$

is a $(k-1)$-dimensional cell locally flatly embedded in the boundary of each of the Q_i.

a) *If $n \geq 5$, $k \neq n-2$, then $Q = Q_1 \cup Q_2$ is a locally flat cell.*

b) *If $n \geq 5$, $k = n-2$, then Q is locally flatly embedded in E^n if and only if $E^n \setminus Q$ has the homotopy type of a circle at every point inside Q^{k-1}.*

Remark 3. Examples for any n, when the union of two locally flat cells of dimension $n-2$ has one singular point, were given by Sosinskii ([4]). The case $n = 3$, $k = 2$ is resolved by Doyle' s theorem ([12]). The case $n = 3$, $k = 1$, see, for example, ([13]), p. 184. Our criterion is also suitable in this case (see also ([14])).

3. Lemma. *Let $s : E^n_+ \to E^n_+$ be a homeomorphism of the closed half-space E^n_+ onto itself, identical on the $(k-1)$-dimensional hyperplane $T \subset E^{n-1} = E^n_+$, and let ∂ be a $(k-1)$-dimensional disk $B^{k-1} \subset T$.*

a) *If $n \geq 5$, $k \neq n-2$, then one can construct a homeomorphism $e : E^n \to E^n$, identical on E^n_-, and such that $(\mathrm{Int}\, esE^n_+) \cup T$ contains some k-dimensional ball segment B^k_+, resting on B^{k-1} and orthogonal to E^{n-1}.*

b) *If $n \geq 5$, $k = n-2$, then the same is true if and only if, at every interior point of B^{k-1}, $E^n_+ \setminus sB^k_+$ has the homotopy type of a circle.*

In proving this lemma (cf. the lemma in ([1])) we substantially use Stallings' technique (see his "engulfing lemma" in ([8, 15])). We give here a brief sketch of the proof.

Denote by $\mathcal{E}_-$ the neighborhood of E^n_- consisting of points whose distance to E^n_- is less than to the k-dimensional half-space E^k_+, bounded by T and orthogonal to E^{n-1}. We do not include T in $\mathcal{E}_-$. Without loss of generality, suppose that $sE^n_+ \subset E^n_+ \setminus \mathcal{E}_-$. Denote by C^n an n-dimensional cube lying in E^n_+ in such a way that one of its faces C^{n-1} lies on E^{n-1} and contains B^{k-1}.

Let first $k \le n-3$. Put $P = [C^n \cap (E^n \setminus \mathcal{E}_-)]$. Fix a triangulation of P, subdividing near T, and let P^2 be the 2-dimensional skeleton of P, and P_*^{n-3} the $(n-3)$-dimensional skeleton of the dual subdivision. It is easy to construct a homeomorphism $e_1 : E^n \to E^n$, identical on E^n_-, such that $e_1\mathcal{E}_- \supset \mathcal{E}_- \cup P^2$. Indeed, $\dim T \le n-4$ and, consequently, the pair $(E^n_+ \setminus T, \mathcal{E}_-)$ is 2-connected. Further, the pair $(E^n_+ \setminus \mathcal{E}_- \setminus T, \operatorname{Int} sE^n_+)$ is homotopically n-connected, and therefore one can construct a homeomorphism $e_2 : E^n \to E^n$, identical on E^n_-, such that $e_2(\operatorname{Int} sE^n) \supset P_*^{n-3}$. The rest of the proof proceeds as in the proof of Stallings' engulfing lemma. The difference is that one has to engulf an infinite polyhedron and therefore, in constructing the homeomorphisms, one has to take care of convergence. As a result the image sE^n_+ will contain, possibly, not the whole polyhedron P, but in any case some neighborhood of B^{k-1} in $E^n_+ \setminus \mathcal{E}_-$. This is evidently sufficient for the proof of the lemma in the case under consideration.

In the case $k \ge n-2$ consider the pair $(E^n_+ \setminus sE_+, \mathcal{E}_-)$. It turns out to be 2-connected if $k = n-2$, under the restriction formulated in the condition of the lemma. In the case $k = n-1$ it splits into two n-connected pairs, since sE^n_+ divides E^n_+. Hence in both cases we can construct e_1. The homeomorphism e_2 is constructed as above.

4. Sketch of the proof of Theorem 2. Since Q_1 is a locally flat cell and since Q^{k-1} lies locally flatly in the boundary of Q_1, one can construct a homeomorphism of E^n onto itself, carrying Q_1 into the half-space $B^k_- \subset E^n_-$ so that $B^k_- \cap E^{n-1}$ is the ball B^{k-1} and is the image of Q^{k-1}. Then one can construct a homeomorphism, identical on B^k_-, which carries the image Q_2 into E^n_+. We shall retain its designation Q_2 for the image Q_2. $Q_2 \cap B_- = B^{k-1}$. Since Q_2 is a locally flat cell and since B^{k-1} lies locally flatly in the boundary of Q_2, it is not hard to construct a homeomorphism $s : E^n_+ \to E^n_+$, identical on the $(k-1)$-dimensional hyperplane T, containing B^{k-1}, and such that $sB^k_+ = Q_2$, where B^k_+ is the k-dimensional half-ball lying in E^n_+, orthogonal to E^{n-1} and resting on B^{k-1}. Now one can apply the lemma and, consequently, apply the method indicated in the proof of the main theorem in ([1]) and fully set forth in ([2]). Therefore one can alter the homeomorphism s so that it has property C from ([1], [2]), i.e. so that

the image of each horn R_t, $|t| \ge 0$, with edge T and axis the orthogonal E^{n-1}, touches R_t. Let v be the center of B^k_+ and let K be the cone vB^{k-1} ($K \subset B^k_+$). As in §3 of (1), it is easy to construct a homeomorphism $g : E^n \setminus B^{k-1} \to E^n \setminus K$, identical on the complement of sE^n_+, and having the property that a sequence Π of points outside K converges to a point inside K if and only if $g^{-1}\Pi$ converges to the projection of this point on B^{k-1} and, moreover, touches a certain horn. The homeomorphism $\bar{s} = gsg^{-1}$ is defined on $E^n_+ \setminus K$, and its image is $sE^n_+ \setminus K$. It can be extended identically to K, owing to the indicated properties of g and s. Moreover, it coincides with s on E^{n-1}, since g is identical on sE^{n-1}. Hence the homeomorphism $h = s\bar{s}^{-1}$ can be extended identically outside sE^n_+. We note that it carries $B_- \cup Q_2$ into $B_- \cup \bar{s}B_+$. Thus it remains only to show that

$B_- \cup \bar{s}B_+$ is locally flat. We shall show that $B_- \cup \bar{s}B_+$ is equivalent to $B_- \cup K$. Construct a homeomorphism $r : E^n_+ \to E^n_+$, identical on E^{n-1}, which carries B_+ onto K. Consider the homeomorphism $\tilde{h} = \bar{s}r\bar{s}^{-1}$. It maps $\bar{s}E^n_+$ onto itself and can be extended identically outside $\bar{s}E^n_+$. Since $\bar{s}$ is identical on K, $\tilde{h}$ carries $B_- \cup \bar{s}B_+$ onto $B_- \cup K$, as required.

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Note: Figure translations are in progress. See original paper for figures.