Academician V. V. SHULEIKIN

1966

SovietRxiv

Fig. 1

Figure 1: Fig. 1

Abstract

Full Text

GEOPHYSICS

Academician V. V. SHULEIKIN

ON THE THEORY OF STEEP WAVES

The theoretical profile of a limiting steep wave not subject to the action of wind was calculated at the end of the last century by Michell ([1]) in accordance with Stokes' ideas ([2]). However, not only was there no rigorous proof of the stable existence of such a wave ([3]), but for a long time even the kinematic nature of the sharpening of the crests of steep waves, up to the appearance of an angular point on their profile, was unknown. In works ([4–7]) we revealed the elements of the kinematics of steep waves, including limiting ones, found the physical meaning of the limiting conditions, and wrote down equations of a family of curves that made it possible to include even wind waves in the investigation:

$$x = R\theta + a\sin\theta, \qquad y = b\cos\theta. \tag{1}$$

Here $R = \lambda/2\pi$; a is the horizontal and b the vertical semiaxis of a certain ellipse that arises instead of the classical circles even for waves in the ocean at finite wave steepness. It turned out that the profile of dead-swell waves of finite steepness differs from a trochoid because the Stokes flow, inseparably connected with potential wave motion, has not a constant velocity, as was previously assumed, but one pulsating within the wave period. The velocity reaches twice the Stokes value at the wave trough and becomes zero at the crest. As the steepness increases,

Fig. 1

this velocity and the amplitude of its oscillations increase. At the limiting—Michell—steepness of the swell, $h/\lambda = 1/7$, the mean velocity of the Stokes flow and the amplitude of its oscillations reach 0.45 of the velocity of the orbital motion of particles on the wave. As a result, it turns out that, relative to a coordinate system moving in the direction of the waves with the mean Stokes velocity, the particles move along ellipses with semiaxes a, b, with $a/b = 1 + b/R$. The minor semiaxis $b_0 = h/2$ at the sea surface.

In Fig. 1 the Michell–Stokes limiting wave is drawn with a dashed line, and the wave satisfying our equations (1) with an almost identical profile is drawn as a

Fig. 2

Figure 2: Fig. 2

solid curve: its steepness is also equal to 1/7, and the crest and trough cut off on the mean line identical segments with the ratio 0.38 : 1. The semiaxes are related to the wavelength by the ratios $a/\lambda = 0.112$; $b/\lambda = 0.0715$. As we see, the differences between the curves in Fig. 1 are very small: the approximate equations (1) did not give an angular point at the crest of the solid curve, where simply a large curvature arose; the tangents to the solid curve at the points of greatest curvature form an angle of 116° with each other instead of the Stokes-Michell value of 120°. After the appearance of our works ([4–7]), unfortunately, no one introduced any refinements into the kinematics of waves, but on

On the basis of Fig. 1 we may assume that the classical condition on the wave surface, satisfied on the basis of the investigations of Stokes and Michell—the condition of constancy of the static pressure along the profile of the dashed curve in Fig. 1—must also be satisfied along the profile of the solid curve with sufficient approximation, the differences between the curves being quite negligible.

All our conclusions in (4-7) were drawn from the continuity condition, written as applied to motion along the orbits of particles on one vertical, between which an entirely different number of water particles may fit in different phases of the process. It was shown that wind waves

Fig. 2

become sharpened at the crests not only because of this effect, but also because of the pulsation, discovered by us, of the velocity of the drift current, which reaches its maximum velocity at the troughs and its minimum at the crests of the waves.

Figure 2 shows three wave profiles with kinematic schemes explaining their origin: *a*—a purely trochoidal profile, which could arise only in the absence of a Stokes current; here, according to the old scheme, water particles move in circles about fixed centers. In Fig. 2*b* a real profile of a wind wave is shown, referred to a coordinate system that moves with velocity $\bar{w}+\bar{u}$ in the direction of the waves, where $\bar{w}$ denotes the averaged velocity of the Stokes current, and $\bar{u}$ the averaged velocity of the drift current. The points on the abscissa axis indicate successive positions of the instantaneous centers of rotation of the water particles in the moving coordinate system; the points on the profiles and on the orbits indicate successive positions of the particles. The ratio of the semiaxes of the ellipse here, on the basis of (7), is as follows:

$$\frac{a_0}{b_0} = 1 + \frac{r_0}{R}\left(1 + \frac{\bar{u}}{v}\right). \tag{2}$$

Here r_0 denotes the radius of rotation, equal to b_0 and $h/2$; v is the velocity of the orbital motion of the particles. In the present case $a_0/b_0 = 1.75$. Figure 2c shows the limiting sharpened profile at the same wave steepness $h/\lambda = 0.12$ as in the two preceding cases. Under the action of the wind, the amplitude of oscillation of the center of rotation of the particles in the moving coordinate system has now increased and has led to an increase of the ratio of the semiaxes to $a_0/b_0 = 2.0$. If the strengthened wind further increases the velocity of the drift current $\bar{u}$ and its pulsation, the profile will become unstable, will pass···

through a purely abstract form with a return point; it would have to acquire, above the crest, a loop resembling Descartes' folium, whereas in reality the wave crests break up into foamy "lambs," while retaining symmetry with respect to the vertical straight lines passing through the crests.

The profiles in Figs. 2 and 2 , which agree well with the results of domestic and foreign investigations in the ocean, and also with investigations in the storm basin of the Marine Hydrophysical Institute, naturally cannot satisfy the linear equations of motion (just as the Stokes–Michell profile cannot). But in the present article we shall not confine ourselves to the assertion that a constant static pressure is preserved along the profile, but shall try, with some approximation, to find: according to what law should the sizes of the orbits of water particles decrease with depth?

It is easy to show first of all that equations (1) are equivalent to the equations

$$x - \alpha = a \sin\theta, \qquad y - \beta = b\cos\theta, \tag{3}$$

in which α and β are the Lagrangian coordinates of the particles at the initial moment of time. Consequently, $\theta = k\alpha - \omega t$, where $k = 2\pi/\lambda$, $\omega = 2\pi/T$; T is the wave period; β is the coordinate of the initial position of the particle, measured downward from the zero level. We write the condition of incompressibility in Lagrangian form:

$$\frac{\partial}{\partial t}\left(\frac{\partial x}{\partial \alpha}\frac{\partial y}{\partial \beta} - \frac{\partial x}{\partial \beta}\frac{\partial y}{\partial \alpha}\right) = 0. \tag{4}$$

Substituting here the partial derivatives with respect to α and to β, found on the basis of the above expressions, we obtain the condition

$$1 + \left(\frac{\partial b}{\partial \beta} + ka\right)\cos\theta + ka\frac{\partial b}{\partial \beta}\cos^2\theta + kb\frac{\partial a}{\partial \beta}\sin^2\theta = \text{const}, \tag{5}$$

which is satisfied only when, simultaneously,

$$\partial b/\partial\beta + ka = 0$$

and, at the same time,

$$a\,\partial b/\partial\beta = b\,\partial a/\partial\beta. \tag{6}$$

At an arbitrary distance β below the zero level, instead of (2), the equation

$$a/b = 1 + nb/R, \tag{7}$$

must be satisfied, in which n denotes, in abbreviated form, the quantity in parentheses in (2). In the case of very steep waves one cannot expect simultaneous satisfaction of three conditions—two (6) and (7)—by two functions a and b. Therefore let us test what follows from comparing them pairwise. First of all, comparison of the two equations (6) gives

$$a = a_0 e^{-\frac{a}{b}k\beta}, \qquad b = b_0 e^{-\frac{a}{b}k\beta}. \tag{8}$$

Comparison of the first of equations (6) with (7) leads to the expression

$$\partial b/\partial\beta + kb(1 + b/R) = 0. \tag{9}$$

Integration of this equation gives

$$b = b_0 \Big/ \left[e^{k\beta} + \frac{b}{R}\left(e^{k\beta} - 1\right)\right]. \tag{10}$$

Using expression (7) once again, one can find a, multiplying b by the right-hand side of (7).

Both expression (8) and expression (10) show that the sizes of the orbits described by water particles decrease with depth more rapidly than followed from the theory of trochoidal waves. Such a phenomenon was observed

by various authors who investigated real waves with sharpened crests ([8],[9]), but it did not receive an exhaustive explanation. We are now convinced that the cause of the accelerated attenuation of the wave motion at depths is connected with the ellipticity of the particle orbits in sharpened waves. Variant (8) does not give a quantitative explanation of the process both because the ellipses here decrease while preserving similarity, and because the factor a/b multiplying $k\beta$ in the exponential function requires an excessively rapid attenuation. Closer to the truth is expression (10), which agrees well with the results of observations set forth in ([9]). Indeed, we note that for $e^{k\beta} \gg 1$ the expression tends in the limit to the form

$$b = b_0 e^{-k\beta}/(1 + b_0/R). \tag{11}$$

Here in the numerator of the fraction on the right-hand side there stands the expression for the radius r of the orbit at depth β, following from the theory of trochoidal waves. In the denominator there stands a quantity equal to the ratio of the semiaxes a_0/b_0 of the orbits at the sea surface. In the case of Michel' s limiting steep sharp-crested wave $a_0/b_0 = 1.45$. In the case shown in Fig. 2, this quantity for a wind wave was equal to 2.0.

Thus the minor semiaxes must decrease at depths by a factor of 1.45–2.0 more than the radii of the circles decrease according to the theory of trochoidal waves. But precisely such ratios were obtained in work ([9]): with the unavoidable scatter of points in the diagrams, the order of the mean values corresponds to our approximate expression. We note also that, for an unlimited decrease of b_0 in comparison with R, (10) becomes the classical relation derived for very gentle waves*.

From the point of view of kinematics, one feature of steep potential waves remains unclear up to the present time. It is known that, according to Stokes, the velocity of the translational motion of water particles associated with a wave of finite steepness can be represented in our notation as

$$\bar{w} = (b/R)^2 c, \tag{12}$$

where c is the phase velocity of gentle waves. Since the minor semiaxis b of the ellipse at a certain depth enters (12) squared, the velocity of the Stokes current decreases with depth considerably faster than b decreases. In full agreement with this, on the basis of (2), the ratio a/b approaches unity, i.e., the eccentricity of the elliptical orbits decreases with depth. On the other hand, the velocity of a steep wave c_1, according to Stokes, is expressed in our notation as

$$c_1 = c\sqrt{1 + (b_0/R)^2} \approx c + {}^1\!/_2 c\,(b_0/R)^2, \tag{13}$$

i.e., approximately $c_1 = c + \bar{w}_0/2$, where $\bar{w}_0$ denotes the averaged velocity of the Stokes current at the sea surface.

Black Sea Division of the Marine
Hydrophysical Institute
Academy of Sciences of the Ukrainian SSR

Received
6 VII 1966

REFERENCES

1. J. Michell, Phil. Mag. (5), **36**, 430 (1893).

2. G. Stokes, Math. and Phys. Papers, **1**, 197, 314 (0000).

3. L. I. Sretenskii, *Theory of Wave Motions of a Liquid*, Moscow, 1936, p. 140.

4. V. V. Shuleikin, DAN, **43**, No. 2, 265 (1953).

5. V. V. Shuleikin, DAN, **45**, No. 5, 987 (1954).

6. V. V. Shuleikin, DAN, **46**, No. 6, 1186 (1957).

7. V. V. Shuleikin, DAN, **113**, No. 5, 1043 (1957).

8. V. V. Shuleikin, DAN, **115**, No. 5, 915 (1957).

9. L. A. Korneva, Izv. AN SSSR, geophys. ser., No. 12, 1815 (1962).

* It would be futile to try to relate the equations of the last remaining pair: the second of equations (6) with equation (7). The set of these equations is satisfied either for a value tending to zero, or for the single value of the quantity, in the case of finite steepness: $n = 0.5$. But such a requirement is impracticable, since always $n \gg 1$.

Note: Figure translations are in progress. See original paper for figures.