ON THE INVARIANT COMPLEMENTABILITY OF SOME SUBSPACES GENERATED BY A LINEAR OPERATOR

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ON THE INVARIANT COMPLEMENTABILITY OF SOME SUBSPACES GENERATED BY A LINEAR OPERATOR

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Let X be a vector space; A a linear operator mapping X into X; N_A the set of all null elements of the operator A $\left(N_A = \bigcup_{n=1}^{\infty} Z_{A^n},\ \text{where } Z_{A^n} = \{x \mid A^n x = \theta\}\right)$. We shall call the *height* $h(x)$ of an element $x \in N_A$ the least of the numbers n for which $A^n x = \theta$. Put $H_A = \sup\{h(x) \mid x \in N_A\}$.

Theorem 1. *If $H_A < \infty$, then in X there exists an A-invariant complement to N_A.*

Proof will be carried out by induction on the magnitude H_A. The case where $H_A = 0$ is trivial. Suppose that for $H_A = m$ the theorem is valid, and let $H_A = m+1$. Put $N'_A = N_A \cap A(X)$, $N''_A = N_A \ominus N'_A$, and denote by X_1 a complement to N''_A in X containing $A(X)$. Let A_1 be the restriction of the operator A to X_1; then $N_{A_1} = N_A \cap X_1 = N'_A$. Since $H_A = m+1$, it follows that $H_{A_1} = m$. By the induction hypothesis, there exists an A_1-invariant complement to N_{A_1} in X_1. This complement will also be an A-invariant complement to N_A in X.

Corollary. *If* $\dim N_A < \infty$, *then in X there exists an A-invariant complement to N_A.*

Denote by $\mathfrak{A}(x_0)$ the linear hull of the set $\{x_0, Ax_0, \ldots, \ldots, A^n x_0, \ldots\}$, $x_0 \in X$.

Theorem 2. *Let $H_A < \infty$. If $x_0 \in N_A$ and $h(x_0) = H_A$, then in X there exists an A-invariant complement to $\mathfrak{A}(x_0)$.*

The proof of this theorem is similar to the proof of Theorem 1.

Theorem 3. *If X_1 is a subspace in N_A and $A(X_1) = X_1$, then in X there exists an A-invariant complement to X_1.*

The proof of this theorem is based on the following lemmas.

Lemma 1. *If X_1 is a subspace in N_A and $A(X_1) = X_1$, then the equation*

$$\sum_{k=1}^{n} \alpha_k A^k x = y,$$

where $y \in X_1$, *has a solution in* X_1.

Proof. The case where $y = \theta$ is trivial. Let $y \neq \theta$. If $\alpha_0 \neq 0$, then the required solution can be found in the form

$$x = \sum_{i=0}^{m} \beta_i A^i y,$$

where $m = h(y) - 1$. If $\alpha_0 = \alpha_1 = \cdots = \alpha_{j-1}$, $\alpha_j \neq 0$, then represent our equation in the form $A^j \left(\sum_{k=j}^{n} \alpha_k A^{k-j} x\right) = y$. Since $A(X_1) = X_1$, there is a $y_0 \in X_1$ such that $A^j y_0 = y$. We arrive at the equation

$$\sum_{k=0}^{n-j} \alpha'_k A^k x = y,$$

where $\alpha'_k = \alpha_{k+j}$. This equation has a solution in X_1, since $\alpha'_0 \neq 0$. The lemma is proved.

Define on $\mathfrak{A}(x_0)$ the function f_{x_0}, taking $f_{x_0}(y)$ to be the least of the numbers p occurring in all possible representations of the element y in the form

$$y = \sum_{k=0}^{p} \alpha_k A^k x_0.$$

Lemma 2. Let $\mathfrak{B}$ be some A-invariant subspace in $\mathfrak{A}(x_0)$. If $y_0 \in \mathfrak{B}$ and

$$f_{x_0}(y_0) = \min\{f_{x_0}(y) \mid y \in \mathfrak{B}\},$$

then $\mathfrak{B} = \mathfrak{A}(y_0)$.

Proof. Obviously, $\mathfrak{A}(y_0) \subset \mathfrak{B}$. We prove the inclusion $\mathfrak{B} \subset \mathfrak{A}(y_0)$. Let

$$y_0 = \sum_{i=0}^{n} \alpha_i A^i x_0, \quad \alpha_n \neq 0, \qquad y = \sum_{k=0}^{m} \beta_k A^k x_0, \quad \beta_m \neq 0, \quad y \in \mathfrak{B}.$$

By the condition of the lemma, $m \geq n$. It is required to show that $y \in \mathfrak{A}(y_0)$. We carry out the proof by induction on the quantity $k(y) = m - n$. If $k(y) = 0$, then $y = \lambda y_0 \in \mathfrak{A}(y_0)$. Suppose that $k(y) = l + 1$, and suppose the inclusion $z \in \mathfrak{A}(y_0)$ is true each time $z \in \mathfrak{B}$ and $0 \leq k(z) \leq l$.

Take

$$z = y - \frac{\beta_m}{\alpha_n} A^{m-n} y_0.$$

It is clear that $z \in \mathfrak{B}$ and $0 \le k(z) \le l$; consequently, $z \in \mathfrak{A}(y_0)$, and

$$y = z + \frac{\beta_m}{\alpha_n} A^{m-n} y_0 \in \mathfrak{A}(y_0).$$

The lemma is proved.

Lemma 3. Let X_1, X_2 be disjoint subspaces in X; $X_0 = X_1 \oplus X_2$; Q_1 and Q_2 are the projection operators generated by the decomposition of X_0 into the direct sum of the subspaces X_1 and X_2 (Q_1 and Q_2 are defined on X_0, $Q_1(X_0) = X_1$, $Q_2(X_0) = X_2$).

For any $E \subset X$ the equality holds

$$Q_1(X_0 \cap E) = (X_2 + E) \cap X_1 \quad (Q_2(X_0 \cap E) = (X_1 + E) \cap X_2).$$

Lemma 4. Let X_0, X_1, X_2, Q_1, Q_2 denote the same objects as in Lemma 3. If X_1, X_2 are A-invariant subspaces, then, whatever the element $x_0 \in X$, the equality

$$\mathfrak{A}(Q_1 v_0) = (X_2 + \mathfrak{A}(u_0)) \cap X_1 \quad (\mathfrak{A}(Q_2 v_0) = (X_1 + \mathfrak{A}(u_0)) \cap X_2),$$

holds, where v_0 is some element of $X_0 \cap \mathfrak{A}(u_0)$ for which

$$f_{u_0}(v_0) = \min\{f_{u_0}(v) \mid v \in X_0 \cap \mathfrak{A}(u_0)\}.$$

Proof. Putting $E = \mathfrak{A}(u_0)$ in Lemma 3, we obtain

$$Q_1(X_0 \cap \mathfrak{A}(u_0)) = (X_2 + \mathfrak{A}(u_0)) \cap X_1.$$

Since the space $X_0 \cap \mathfrak{A}(u_0)$ is A-invariant, by Lemma 2,

$$X_0 \cap \mathfrak{A}(u_0) = \mathfrak{A}(v_0).$$

Consequently,

$$Q_1(\mathfrak{A}(v_0)) = (X_2 + \mathfrak{A}(u_0)) \cap X_1.$$

To complete the proof it remains to take into account the equality

$$Q_1(\mathfrak{A}(v_0)) = \mathfrak{A}(Q_1 v_0),$$

which follows from the commutativity on X_0 of the operators A and Q_1.

Proof of Theorem 3. Let X_2 be a maximal A-invariant subspace in X, disjoint from X_1. Suppose that $X_0 = X_1 \oplus X_2 \neq X$. Let $x_0 \in X_0$. Choose in $X_0 \cap \mathfrak{A}(x_0)$ an element y_0 satisfying the condition

$$f_{x_0}(y_0) = \min\{f_{x_0}(y) \mid y \in X_0 \cap \mathfrak{A}(x_0)\},$$

and denote by $P_0(A)$ the polynomial in A for which

$$P_0(A) x_0 = y_0.$$

By Lemma 1, there exists an element $z_0 \in X_1$ such that

$$P_0(A)z_0 = Q_1 y_0.$$

Put

$$u_0 = x_0 - z_0, \qquad v_0 = P_0(A)u_0.$$

It is easy to see that

$$f_{u_0}(v_0) = \min\{f_{u_0}(v) \mid v \in X_0 \cap \mathfrak{A}(u_0)\}.$$

By Lemma 4, we have the equality

$$(X_2 + \mathfrak{A}(u_0)) \cap X_1 = \mathfrak{A}(Q_1 v_0),$$

from which it follows that

$$(X_2 + \mathfrak{A}(u_0)) \cap X_1 = \{\theta\}$$

(for $v_0 = y_0 - Q_1 y_0$). We have arrived at a contradiction, since $X_2 + \mathfrak{A}(u_0) \neq X_2$. The theorem is proved.

Corollary. Let

$$M_A = \bigcap_{n=1}^{\infty} A^n(X).$$

If $A(M_A) = M_A$, then in X there exists an A-invariant complement to $M_A \cap N_A$.

Theorem 4. If, beginning with some number m, the sets

$$Z_n = Z_A \cap A^n(X), \quad n = 0, 1, \dots,$$

coincide with one another, then in X there exists an A-invariant complement to N_A.

Proof. It is known ([1], Theorem 1) that, under the hypotheses of the theorem being proved, $A(M_A) = M_A$. Hence, by the corollary to Theorem 3, there exists in X an A-invariant complement X_1 to $M_A \cap N_A$. From these same hypotheses it is easy to conclude that

$$\sup\{h(x) \mid x \in N_A \cap X_1\} = m.$$

Therefore, denoting by A_1 the restriction of the operator A to X_1, we have $H_{A_1} = m < \infty$. By Theorem 1, there exists in X_1 an A_1-invariant complement X_2 to

$$N_{A_1} = N_A \cap X_1.$$

Obviously, X_2 will also be an A-invariant complement in X to N_A.

Corollary. *If at least one of the numbers* $\dim Z_A$ *or* $\operatorname{codim} A(X)$ *is finite, then there exists in* X *an* A*-invariant complement to* N_A *(for from these conditions the hypotheses of Theorem 4 follow (see [1], Theorem 2)).*

We give an example of an operator A for whose null elements there does not exist an A-invariant complement in X.

Consider the vector space X with basis

$$a_{ij}, \quad b_k, \quad c_m; \qquad i, j, k = 1, 2, \dots; \quad j \le i; \quad m = 0, 1, \dots .$$

Define on X a linear operator A by setting

$$Aa_{i1} = \theta \quad (i = 1, 2, \dots); \qquad Aa_{ij} = a_{i,j-1} \quad (i = 1, 2, \dots;\ 2 \le j \le i);$$

$$Ab_k = b_{k+1} \quad (k = 1, 2, \dots); \qquad Ac_0 = b_1; \qquad Ac_m = c_{m-1} + a_{mm} \quad (m = 1, 2, \dots).$$

The formulas

$$A^{m+1} c_m = b_1 \quad (m = 0, 1, \dots), \qquad A^m c_{m+1} = c_1 + \sum_{i=2}^{m+1} a_{i2} \quad (m = 1, 2, \dots)$$

are valid. It is easy to see that $a_{ij} \in N_A$. We shall show that the elements a_{ij} form a basis in N_A. Let $x \in N_A$ and

$$x = \sum_{i=1}^{r} \sum_{j=1}^{i} \alpha_{ij} a_{ij} + \sum_{k=1}^{s} \beta_k b_k + \sum_{m=0}^{t} \gamma_m c_m .$$

Choose $n > \max\{r, t\}$ such that $A^n x = \theta$. Applying the operator A^n to the element x, we obtain

$$\theta = \sum_{k=1}^{s} \beta_k b_{k+n} + \sum_{m=0}^{t} \gamma_m b_{n-m} .$$

It follows that

$$\beta_k = \gamma_m = 0 \quad (k = 1, 2, \dots, s;\ m = 1, 2, \dots, t).$$

Suppose that there exists an A-invariant complement F to N_A. We shall show that

$$c_m \in F \quad (m = 1, 2, \dots).$$

For this, in the equality

$$Ac_{m+1} = c_m + a_{m+1,m+1}$$

replace c_{m+1} by the sum $c'_{m+1} + c''_{m+1}$, where

$$c'_{m+1} \in N_A, \qquad c''_{m+1} \in F;$$

we obtain the equality

$$Ac''_{m+1} - c_m = -Ac'_{m+1} + a_{m+1,m+1} .$$

Since the equation

$$Ax = a_{m+1,m+1}$$

is insoluble, we have

$$\theta \neq -Ac'_{m+1} + a_{m+1,m+1}.$$

Assuming that $c_m \in F$, we arrive at the relation

$$\theta \neq (Ac''_{m+1} - c_m) \in N_A \cap F,$$

contradicting the equality

$$N_A \cap F = \{\theta\}.$$

Thus, $c_m \in F$. In particular,

$$c'_1 \neq \theta.$$

Let

$$c'_1 = \sum_{i=1}^{r} \sum_{j=1}^{i} \alpha_{ij} a_{ij}.$$

Write the equalities

$$A^r c''_{r+1} = A^r c_{r+1} - A^r c'_{r+1} = c_1 + \sum_{i=2}^{r+1} a_{i2} - A^r c'_{r+1},$$

from which it follows that

$$A^r c''_{r+1} - c''_1 = c'_1 - A^r c'_{r+1} + \sum_{i=2}^{r+1} a_{i2}.$$

The obtained equality is contradictory, for

$$A^r c''_{r+1} - c''_1 \in F, \qquad c'_1 - A^r c'_{r+1} + \sum_{i=2}^{r+1} a_{i2} \in N_A,$$

and

$$c'_1 - A^r c'_{r+1} + \sum_{i=2}^{r} a_{i2} \neq 0.$$

The latter follows from the fact that the element $a_{r+1,2}$, which appears in the sum

$$\sum_{i=2}^{r+1} a_{i2},$$

does not occur in the expansions with respect to the basis a_{ij} of the null elements c'_1 and $A^r c'_{r+1}$. This proves that there is no A-invariant complement to N_A.

The example constructed permits us to assert that not every maximal A-invariant subspace disjoint from N_A is a complement to N_A in X. One can show that this assertion remains valid even in the case when

$$\dim N_A < \infty.$$

However, the following theorem holds.

Theorem 5. *If A is a locally algebraic operator, then every maximal A-invariant subspace disjoint from N_A is a complement to N_A.*

(The operator A is called locally algebraic if, for every $x \in X$, the subspace $\mathfrak{A}(x)$ is finite-dimensional.)

The proof of this theorem follows easily from the following lemma.

Lemma 5. *Let E be some invariant subspace disjoint from N_A. If $x_0 \in E \oplus N_A$ and $\mathfrak{A}(x_0)$ is a finite-dimensional subspace, then there exists an A-invariant subspace $F \supset E$ such that $F \cap N_A = \{\theta\}$ and $x_0 \in F \oplus N_A$.*

Proof. Let L be an A-invariant complement to $N_A \cap \mathfrak{A}(x_0)$ in $\mathfrak{A}(x_0)$. Since L contains no zeros of the operator A (except θ), it follows that $A(L) = L$.

It is clear that $E \dot{+} L (= F)$, and F is an A-invariant subspace. We show that $N_A \cap F = \{\theta\}$. To this end, note that the set $L \setminus E$ is invariant with respect to A (by the invertibility of A on L and the A-invariance of $L \cap E$). Let

$$y \in N_A \cap F = N_A \cap \{E + (L \setminus E)\}.$$

Then there exists an n such that $A^n y = \theta$. Represent the element y in the form $y = y' + y''$, where $y' \in E$, $y'' \in (L \setminus E) \cup \{\theta\}$. Since $A^n y' \in E$, $A^n y'' \in (L \setminus E) \cup \{\theta\}$, and $A^n y' + A^n y'' = \theta$, it follows that $y = \theta$.

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REFERENCES

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Note: Figure translations are in progress. See original paper for figures.