ON THE NORMAL STRUCTURE OF FINITE GROUPS

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§ 1. The appearance of the works of P. Hall and G. Higman ([1]) and H. Wielandt ([2]) aroused new interest in the study of the normal structure of finite groups. Both p-solvable groups, introduced by S. A. Chunikhin ([3]), and non-p-solvable groups with prescribed properties of subgroups were investigated. Along these lines, classes of finite groups with p-length equal to one were found ([4], [5]) (by the p-length of a finite group we mean the least number of p-factors occurring in its invariant series, the non-p-factors of which are elementary).

In the present note we give results of an investigation of finite groups with a certain system of cyclic chief factors (factors of a chief series). We study (see § 5) finite groups in which every chief factor has a cyclic Sylow p-subgroup. As it turns out, the p-length of such groups is equal to one. The class of groups under consideration is sufficiently broad: it includes p-supersolvable groups and groups with a cyclic Sylow p-subgroup. In view of this, from the results obtained there follow Theorem XVII of S. A. Chunikhin ([6]) and Theorem 4.2 of H. Wielandt ([2]).

In § 4 results are given on the characterization of finite groups with a system of cyclic chief factors by means of the Frattini subgroup, the generalized Fitting subgroup, and the indices of maximal subgroups. The basis for obtaining these results is formed by Theorems 1 and 2, which pertain to the theory of modular representations and are of independent interest.

In § 6 a new D-theorem is formulated, from which follows Theorem D5 of P. Hall ([14]).

Notation: Π is some set of prime numbers; p is a fixed prime number; Π' is the complement of Π in the set of all prime numbers; if $\Pi = \{p\}$, then $p' = \Pi'$; G is a finite group, G_p is one of its Sylow p-subgroups of order $(G)_p$; $\Phi(G)$ is the Frattini subgroup of the group G; $F_p(G)$ is the largest p-nilpotent normal divisor of the group G; $F_\Pi(G)$ is the intersection of all subgroups $F_p(G)$ for distinct $p \in \Pi$; E is the identity subgroup; $m(G)$ is the minimal number of generators of the group G; $GL(n, F)$ is the group of all nonsingular square matrices of degree n over the field F. For the notions of Π-solvability, Π-supersolvability, and p-nilpotence, see ([7]).

A series of subgroups

$$A = A_0 \supset A_1 \supset \ldots \supset A_t = E$$

of an abelian group A with a domain of operators S is called an S-composition series if each subgroup A_i, $i > 0$, is a proper maximal S-admissible subgroup of the group A_{i-1}. An invariant subgroup N of a group G is called p-supersolvably embedded in G if the indices of the chief series of the group G on the section from N to E are either equal to p, or are not divisible by p. If an invariant subgroup N of G is p-supersolvably embedded in G for every prime number p, then N is called supersolvably embedded in G (see ([8])).

§ 2. **Lemma 1.** Let G be a Π-solvable group. Then $C \subseteq F_\Pi(G)$, where C is the centralizer of the subgroup $F_\Pi(G)$ in the group G.

Lemma 2. Let G be a p-solvable group with p-length equal to one. Then a Sylow p-subgroup of $\Phi(G)$ is the intersection of the Frattini subgroups of all Sylow p-subgroups of the group G.

From Lemma 1 there follows the well-known proposition of Fitting (([9]), p. 107). Lemma 2 generalizes Theorem 5 of ([10]).

§ 3. In the hypotheses of the next two theorems, F will denote some field of characteristic $p > 0$, and X a finite or infinite group. As usual, an F-X-module ([11]) is called **irreducible** if it contains no nontrivial submodules. The dimension of an F-X-module M will be denoted by $\dim_F M$ (only finite-dimensional modules are considered). If $\rho(x)$, $x \in X$, is the representation determined by the F-X-module M, then $\rho(R)$ is the image of the subgroup R of the group X under the homomorphism $\rho : X \to GL(m, F)$.

Theorem 1. Let the F-X-module M contain an irreducible submodule N such that the factor module M/N is irreducible, and let P be some invariant finite p-subgroup of the group X. If

$$\dim_F N > m(P) \dim_F M/N,$$

then P is contained in the kernel of the representation $\rho(x)$ determined by the F-X-module M.

Theorem 2. Let an F-X-module M of dimension m contain an irreducible submodule N of dimension n, which is not a direct summand in M. Let $\rho(x)$ and $\tau(x)$ be the representations of the group X determined respectively by the modules M and M/N. If R is such a finite invariant p'-subgroup of the group X that $\tau(R)$ is contained in the center of $GL(m-n, F)$, then $\rho(R)$ is contained in the center of $GL(m, F)$.

§ 4. **Theorem 3.** Let A be a finite abelian p-group with a p-supersolvable group of operators G. Then there exists a G-composition series

$$A = A_0 \supset \cdots \supset A_s \supset \cdots \supset A_t = E, \qquad 0 \leq s \leq t,$$

such that on the interval from A to A_s there are no indices equal to p, and on the interval from A_s to E there are no indices divisible by p^2.

Theorem 4. Let G contain a normal divisor N, each composition factor of which is either a Π-group or a Π'-group, and suppose that the index in the group G of each of its maximal subgroups containing N is either equal to some prime number from Π', or is not divisible by any prime number from Π. Then N is Π-supersolvable. If, in addition, N contains no invariant Π'-subgroups distinct from E, then $N/N \cap \Phi(G)$ is supersolvably embedded in $G/N \cap \Phi(G)$.

Theorem 5. An invariant subgroup N of a group G is p-supersolvably embedded in G if and only if $N/\Phi(N)$ is p-supersolvably embedded in $G/\Phi(N)$.

Corollary 1. If $G/\Phi(G)$ is p-supersolvable, then G is also p-supersolvable.

Corollary 2. A Π'-solvable Π-group ([13]) is Π-supersolvable.

Theorem 6. Let G contain a Π-solvable normal divisor N having no invariant Π'-subgroups distinct from E. If $F_\Pi(N)$ is supersolvably embedded in G, then the subgroup N also is supersolvably embedded in G.

Corollary. If G contains a p-solvable invariant subgroup N, and $F_p(N)$ is p-supersolvably embedded in G, then N also is p-supersolvably embedded in G.

If $N = G$ and G is a Π-group, then Theorem 4 gives Theorem 9 of ([10]), and from Theorem 6 follows Theorem 13 of ([10]). If

p is an arbitrary prime number, then Theorem 5 yields a result of P. Baer ([8]).

§ 5. Denote by $c_p(G)$ the product of all those indices of the principal series of the group G which are equal to p. If among the indices of the principal series there are none equal to p, put $c_p(G) = 1$.

The object of our study will now be finite groups possessing the following property:

Every principal factor of the group G has a cyclic Sylow p-subgroup. (*)

It is clear that subgroups and factor groups of a group G possessing property $(*)$ also possess this property.

One of the most important properties of the groups under consideration is given by the following theorem.

Theorem 7. *Let K be the commutator subgroup of a group G possessing property $(*)$. Then K contains a characteristic subgroup N such that $(K : N) = c_p(K)$.*

Since p-supersolvable groups possess property $(*)$, Theorem 7 implies a result of S. A. Chunikhin on the p-nilpotency of the commutator subgroup in p-supersolvable groups ([6]).

Theorem 8. *Let G possess property $(*)$. Then there exists a normal divisor N of the group G such that:*

1) *the factor group G/N is p-supersolvable;*
2) *$c_p(G) = (G/N)_p$ and, consequently, N contains no composition factors of order p;*
3) *$G_p \cap N$ has a complement in G_p, i.e.*

$$G_p = [G_p \cap N]P, \quad [G_p \cap N] \cap P = E;$$

4) *the Sylow p-subgroup N_p of N is representable in the form*

$$N_p = P_1 \times P_2 \times \cdots \times P_t,$$

where each P_i is isomorphic to a Sylow subgroup of some principal factor of the group N.

We note that, using Theorem 7, in Theorem 8 we may choose the subgroup N so that N is contained in the commutator subgroup of the group G and G_{pN} is invariant in G. In addition, with the aid of Theorem 1 from ([12]) one can obtain some information about the order of the subgroup N.

A consequence of Theorem 8 is Theorem 4.2 of G. Wielandt ([2]).

§ 6. In conclusion we dwell on the relationship between the Sylow properties of a finite group and the properties of the factors of its invariant series. At present, for denoting a series of Sylow properties the generally accepted notation is that of P. Hall ([14]). We shall introduce one more symbol of a similar kind, using the notion of Π-regularity of P. Baer ([15]).

A group G is called Π-regular if there exists such an ordering φ of the set Π under which the group G and each of its subgroups possesses a φ-dispersive ([7]) S_Π-subgroup.

We shall say that a group G possesses property E_Π^{nr} if G contains an S_Π-subgroup H, representable in the form $H = H_1 \times \cdots \times H_k$, with: 1) the orders of the subgroups H_i and H_j for $i \neq j$ relatively prime; 2) G is a Π_i-regular group, where Π_i is the set of all prime divisors of the number (H_i), $i = 1, 2, \dots, k$.

Theorem 9. *If K is such a normal divisor of the group G that K possesses property E_{Π}^{nr}, and G/K possesses property D_{Π}^{s}, then G possesses property D_{Π}^{s}.*

Corollary. *If every factor of some series of normal divisors of the group G possesses property E_{Π}^{nr}, then G possesses property D_{Π}^{s}.*

We note that Theorem 9 and Theorem A from ([16]) can be combined into one more general result.

§ 7. The proof of Theorem 9 is based on several general propositions, which we shall now formulate.

Let θ and η be certain group-theoretic properties, hereditary for subgroups and factor groups, and let G be a group satisfying the following conditions:

D1. In G there is a normal divisor K possessing the properties D_{Π} and θ.

D2. The factor group G/K possesses the properties D_{Π} and η.

D3. G does not possess the property D_{Π}.

D4. Every nontrivial subgroup and factor group of the group G satisfying conditions D1 and D2 possesses the property D_{Π}.

Then the following propositions hold:

I. G possesses at least one S_{Π}-subgroup H.

II. $G = KH = KA$, where A is any Π-subgroup of F which is not contained in any of the subgroups conjugate to H in G.

III. G has no invariant Π-subgroups distinct from E.

IV. If M is a maximal Π-subgroup of the group G and M is not conjugate to H in G, then G has no nontrivial invariant subgroup R for which the following conditions would simultaneously be fulfilled: a) $R \supseteq K$; b) $R \cap M$ possesses an invariant Hall subgroup L distinct from E; c) $N \cap K$ possesses the property D_{Π}, where N is the normalizer of the subgroup L in G; d) if L is not a Hall subgroup in R, then L is not a Hall subgroup in $N \cap R$; e) $H \cap R$ possesses the property C_w, where w is the set of all prime divisors of the number (L).

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Note: Figure translations are in progress. See original paper for figures.