ON THE GROWTH OF ENTIRE FUNCTIONS OF SEVERAL COMPLEX VARIABLES

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ON THE GROWTH OF ENTIRE FUNCTIONS OF SEVERAL COMPLEX VARIABLES

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In this note we give a number of results on the growth of an entire function of several variables with respect to one of the variables when the values of the others are fixed.* In the case of functions of two variables, questions of this kind were studied earlier in papers (^{7}-^{10}). The results of these authors are extended here to the case of functions of n $(n \geq 2)$ variables, with their simultaneous refinement and strengthening in the case $n = 2$. Results are also obtained that have no analogue among those known previously. The proofs of all the results, in view of their cumbersomeness, are omitted.

We introduce the following notation and definitions.

The space of complex variables $z_1, z_2, \dots, z_k$ will be denoted by C_k.

Let $f(z_1, \dots, z_n, z_{n+1})$ be an entire function in C^{n+1}. Denote

$$\rho_f(z_1, \dots, z_n) = \overline{\lim_{R\to\infty}} \frac{1}{\ln R} \ln\ln \left\{ \max_{|z_{n+1}|=R} |f(z_1, \dots, z_n, z_{n+1})| \right\},$$

$$\rho^*_f(R_1, \dots, R_n) = \overline{\lim_{R_{n+1}\to\infty}} \frac{1}{\ln R_{n+1}} \ln\ln \left\{ \max_{|z_i|=R_i,\, i=1,\dots,n+1} |f(z_1, \dots, z_{n+1})| \right\},$$

$$\overline{\rho} = \sup_{R_i<\infty,\, i=1,\dots,n} \rho^*_f(R_1, \dots, R_n),$$

$$\sigma_{f,\rho}(z_1, \dots, z_n) = \overline{\lim_{R\to\infty}} R^{-\rho} \ln \left\{ \max_{|z_{n+1}|=R} |f(z_1, \dots, z_n, z_{n+1})| \right\},$$

$$\sigma^*_{f,\rho}(R_1, \dots, R_n) = \overline{\lim_{R_{n+1}\to\infty}} R_{n+1}^{-\rho} \ln \left\{ \max_{|z_i|=R_i,\, i=1,\dots,n+1} |f(z_1, \dots, z_{n+1})| \right\}.$$

We shall denote by $C(E)$, where E is some plane set, the inner capacity of the set E.

By $\Delta(E; z_1', \dots, z_{k-1}')$, where $E \subset C^k$, we shall denote the intersection of E with the plane

$$\{(z_1, \dots, z_k) : z_1 = z_1', \dots, z_{k-1} = z_{k-1}',\ |z_k| < \infty\}.$$

For $k > 1$, by $\Gamma_k(E)$, where $E \subset C^k$, we shall denote the set of those points $(z_1, \dots, z_{k-1})$ for which $C(\Delta(E; z_1, \dots, z_{k-1})) > 0$. We also put $\Gamma_1(E) = E$ for $k = 1$, and for $k \geq 1$

$$\Gamma_k^1(E) = \Gamma_1(\Gamma_2(\dots(\Gamma_{k-1}(\Gamma_k(E)))\dots)).$$

A bounded set $E \subset C^k$ will be called **satisfying condition A** if $C(\Gamma_k^1(E)) > 0$.

A set E, lying in C^k and not coinciding with C^k, is called **satisfying condition B** if the intersection of E with any analytic plane $\{z_i = a_i w + b_i,\ i = 1, \dots, k\}$, not lying entirely in E, belongs to F_σ and has zero inner capacity.

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* From other points of view, the growth of entire functions of several variables was studied in papers (^{1}-^{6}) and others.

Let us note that the fulfillment of condition A for a certain set E indicates its "massiveness." Thus, condition A is obviously satisfied by sets having positive Lebesgue measure in the space. Conversely, the fulfillment of condition B for a set E testifies to its "rarity." Thus, in (^{11}) it was indicated that a set satisfying condition B has Lebesgue measure zero in the space C^k. Moreover, it was shown there that the intersection of such a set with the k-dimensional real space $\{\operatorname{Im} z_i = 0,\ i = 1, \dots, k\}$ also has Lebesgue measure zero in this space.

In the case when the growth of an entire function $f(z_1, \dots, z_n, z_{n+1})$ with respect to the variable z_{n+1} is characterized by its orders $\rho_f(z_1, \dots, z_n)$ and $\rho_f^*(R_1, \dots, R_n)$, the following holds.

Theorem 1. If, for an entire function $f(z_1, \dots, z_{n+1})$, the inequality

$$\rho_f(z_1, \dots, z_n) < \infty$$

holds on a set E satisfying condition A, then $\bar{\rho} < \infty$, $\rho_f^*(R_1, \dots, R_n) \equiv \bar{\rho}$, and the set N_ρ of those points $(z_1, \dots, z_n)$ at which the strict inequality

$$\rho_f(z_1, \dots, z_n) < \bar{\rho}$$

holds satisfies condition B.

If one studies the growth of an entire function not in planes parallel to one of the coordinate planes, but in planes passing through the origin of coordinates, then, with the aid of Theorem 1, the following theorem is established.

Theorem 2. Let $f(z_1, \dots, z_n)$ be an entire function. Suppose, further, that the function

$$\varphi(z_1, \dots, z_n, \zeta) = f(z_1\zeta, \dots, z_n\zeta)$$

has, with respect to the variable ζ, on the set

$$\{(z_1, \dots, z_n) : (z_1, \dots, z_{n-1}) \in E,\ \ z_n = 1\},$$

where the set E satisfies condition A, finite order $\rho_\varphi(z_1, \dots, z_n)$. Then the function $f(z_1, \dots, z_n)$ has, in the aggregate of variables, some finite order ρ^*, and

$$\rho_\varphi(z_1, \dots, z_{n-1}, 1) = \rho$$

everywhere in C^{n-1}, with the possible exception of a set satisfying condition B.

We now establish the relation between the growth of a function in the aggregate of variables and its growth in each variable separately. It is known that the order of a function in the aggregate of variables is, generally speaking, greater than its orders in each of the variables separately. Thus, the function e^{zw} has order 1 in each of the variables, while in the aggregate of variables its order is equal to 2. At the same time, it turns out to be possible to estimate from above the order of a function in the aggregate of variables through its orders in each of the variables separately.

Theorem 3. If, for any $i = 1, \dots, n$, the order $\rho_i(z_1, \dots, z_{i-1}, z_{i+1}, \dots, z_n)$ of an entire function $f(z_1, \dots, z_n)$ with respect to the variable z_i satisfies, on a set E_i possessing property A, the condition

$$\rho_i(z_1, \dots, z_{i-1}, z_{i+1}, \dots, z_n) < \infty,$$

then the function $f(z_1, \dots, z_n)$ has, with respect to the aggregate of the variables, some finite order ρ, satisfying the inequality

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* The order of an entire function in the aggregate of variables may be defined by the equality

$$\rho = \lim_{R\to\infty} \frac{1}{\ln R} \ln\ln \left\{ \max_{|z_i|=R,\ i=1,\dots,n} |f(z_1, \dots, z_n)| \right\}$$

(see, for example, (^{12}), § 26).

$$\rho \leq \hat{\rho}_1 + \dots + \hat{\rho}_n,$$

where

$$\hat{\rho}_i = \sup_{E_i} \rho_i(z_1, \dots, z_{i-1}, z_{i+1}, \dots, z_n).$$

From this theorem, in particular, it follows that an entire function which has order not exceeding 1 in each variable has, with respect to the aggregate of the variables, order not exceeding 2.

It is easy to see that a theorem analogous to Theorem 1 does not hold if the growth of a function is characterized not by its orders, but by its types $\sigma_{f,\bar{\rho}}(z_1, \dots, z_n)$ and $\sigma^*_{f,\bar{\rho}}(R_1, \dots, R_n)$. However, in this case as well, if certain additional requirements are imposed on the function, a number of assertions about its growth can be made. Namely, the following theorems hold:

Theorem 4. Let the entire function $f(z_1, \dots, z_{n+1})$ have finite order ρ with respect to the aggregate of the variables and, on some set $E \subset C^n$ possessing property A, satisfy the condition

$$\sigma_{f,\bar{\rho}}(z_1, \dots, z_n) \leq \gamma < \infty.$$

Then, for any $R_i > 0,\ i = 1, \dots, n$,

$$\sigma^*_{f,\bar{\rho}}(R_1, \dots, R_n) \leq C_f \gamma \prod_{i=1}^{n} (R_i + R_i^0)^{\rho *}.$$

where C_f is a constant depending on the choice of the function $f(z_1, \dots, z_{n+1})$, and the numbers R_i^0 are such that $E \subset \{|z_i| < R_i^0,\ i = 1, \dots, n\}$.

Theorem 5. Let the entire function $f(z_1, \dots, z_{n+1})$, for any $R_1 > 0, \dots, R_n > 0$, satisfy the condition

$$\sigma^*_{f,\bar{\rho}}(R_1, \dots, R_n) < \infty.$$

Then $\ln \tilde{\sigma}_{f,\bar{\rho}}(z_1, \dots, z_n)$, where

$$\tilde{\sigma}_{f,\bar{\rho}}(z_1, \dots, z_n) = \lim_{\varepsilon \to 0} \sup_{|z_i - z_i'| < \varepsilon,\ i=1,\dots,n} \sigma_{f,\bar{\rho}}(z_1', \dots, z_n'),$$

is a plurisubharmonic function, and the set M_ρ of points $(z_1, \dots, z_n)$ at which

$$\sigma_{f,\bar{\rho}}(z_1, \dots, z_n) < \tilde{\sigma}_{f,\bar{\rho}}(z_1, \dots, z_n),$$

belongs to F_σ and satisfies the condition

$$C(\Gamma_n^1(M_\rho)) = 0.$$

Without dwelling on the proofs of the theorems stated here, let us only note that we made extensive use of the following lemma, which, in our opinion, is also of independent interest.

Lemma. Let E be a closed planar set lying in the disk $|z| < R_0$ and having positive capacity, and let E^* be a connected component complement to E containing the infinitely distant point. Further, let $\omega(z, E, R)$, where $R > R_0$, be a function harmonic in $E^* \cap (|z| < R)$, taking the value 1 everywhere on the circle $|z| = R$ and 0 at all boundary points of E^*, with the possible exception of some set of capacity zero. Then, for $R > 2R_0$ and $R_0 < |z| < R$, the inequality

$$\omega(z, E, R) \geq 1 - \frac{\ln|z| - \ln C(E)}{\ln R - \ln R_0}$$

holds.

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* It can be shown that, in the absence of restrictions on the growth of the function $f(z_1, \dots, z_{n+1})$ with respect to the aggregate of the variables, the function $\sigma_{f,\bar{\rho}}(z_1, \dots, z_n)$ may be any function whose logarithm is plurisubharmonic.

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Note: Figure translations are in progress. See original paper for figures.