Mathematical Physics

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Abstract

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On the Question of Solving the Kinetic Transport Equation by the Splitting Method

Consider the linear integro-differential equation

$$\vec{\Omega}\nabla\varphi(\mathbf{r},\vec{\Omega}) + \sigma\varphi(\mathbf{r},\vec{\Omega}) = \frac{\sigma_s}{4\pi}\int \varphi(\mathbf{r},\vec{\Omega}')\,d\vec{\Omega}' + f(\mathbf{r},\vec{\Omega}), \tag{1}$$

which describes stationary one-velocity processes of radiation transport in a medium with isotropic scattering. For equation (1) in a bounded and convex domain G with piecewise smooth boundary Γ, we impose the boundary condition

$$\varphi(\mathbf{r},\vec{\Omega}) = 0 \quad \text{on } \Gamma \text{ when } \vec{\Omega}\mathbf{n} < 0. \tag{2}$$

Here $\mathbf{n}$ is the unit vector of the outward normal at the point with radius vector $\mathbf{r}$ of the boundary Γ. We assume that the functions $\sigma = \sigma(\mathbf{r})$, $\sigma_s = \sigma_s(\mathbf{r})$, characterizing the properties of the medium, are positive and piecewise continuous in the domain, i.e.

$$0 < \sigma(\mathbf{r}) \leq \sup_{r\in G}\operatorname{vrai}\sigma(\mathbf{r}) = \sigma^0 < \infty, \tag{3}$$

$$0 < \sigma_s(\mathbf{r}) \leq \sup_{r\in G}\operatorname{vrai}\sigma_s(\mathbf{r}) = \sigma_s^0 < \infty. \tag{4}$$

With respect to the function $f(\mathbf{r},\vec{\Omega})$, which characterizes the intensity of the radiation sources, we assume that it is piecewise continuous in the domain $S\times G$. Here and below S will denote the set of all unit direction vectors $\vec{\Omega}$ in three-dimensional Euclidean space R_3.

Introduce the variable η by the formula

$$\mathbf{r} - \eta\vec{\Omega} = \mathbf{r}'. \tag{5}$$

Then along the ray $\vec{\Omega}$ the kinetic equation (1) and the boundary condition are written, respectively, in the following form:

$$\frac{\partial}{\partial \eta}\varphi(\mathbf{r}'+\eta\vec{\Omega},\vec{\Omega})+\sigma\varphi(\mathbf{r}'+\eta\vec{\Omega},\vec{\Omega}) = \frac{\sigma_s}{4\pi}\int \varphi(\mathbf{r}'+\eta\vec{\Omega},\vec{\Omega}')\,d\vec{\Omega}'+f(\mathbf{r}'+\eta\vec{\Omega},\vec{\Omega}), \quad (1')$$

$$\varphi(\mathbf{r}' + \eta_0\vec{\Omega},\vec{\Omega}) = 0, \quad (2')$$

where η_0 is the value of η at which the radius vector $\mathbf{r}$ intersects the boundary of the domain G.

Formula (5), for each $\vec{\Omega}$, gives a decomposition of the set G into the Cartesian product of the two-dimensional set $\pi_{\vec{\Omega}}$ and the one-dimensional set $\pi_{\vec{\Omega},\mathbf{r}'}$:

$$G = \pi_{\vec{\Omega}} \times \pi_{\vec{\Omega},\mathbf{r}'}. \quad (6)$$

Following (1), let us define a class of real functions D, in which we shall seek the solution of equation (1). We assign to the class D functions φ possessing the following properties: 1) for almost all $(\mathbf{r}',\vec{\Omega})$ from $S \times \pi_{\vec{\Omega}}$

the function $\varphi(\mathbf{r}' + \eta\vec{\Omega},\vec{\Omega})$ is absolutely continuous on the closed set $\overline{\pi_{\Omega,r'}}$; 2) for almost all $(\mathbf{r}',\vec{\Omega})$ from $S \times \pi^{+}_{\Omega}$, the function satisfies the condition $\varphi(\mathbf{r}' + \eta_0\vec{\Omega},\vec{\Omega}) = 0$; 3) the function φ must be such that $l\varphi = \dfrac{\partial}{\partial\eta}\varphi(\mathbf{r}' + \eta\vec{\Omega},\vec{\Omega}) \in \mathcal{L}_2(S \times G)$. The set of functions D is dense in $\mathcal{L}_2(S \times G)$. A proof of this fact is contained in work (^{1}).

Conditions 1)–3) determine the class D of generalized solutions of equation (1). In this case, by a generalized solution of equation (1) we shall mean any function from D satisfying equation (1) almost everywhere.

Introduce the operator

$$\Lambda_2 = \sigma I - \frac{\sigma_s}{4\pi}\int d\vec{\Omega}' \quad (7)$$

and denote by Λ_1 the operator l, whose domain of definition consists of the functions of the set D. Then, taking into account the introduced operators Λ_1, Λ_2, the boundary-value problem (1), (2) can be written in operator form as

$$(\Lambda_1 + \Lambda_2)\varphi = f. \quad (8)$$

We solve equation (1) by means of the splitting method according to the generalized relaxation scheme

$$(I + \tau\Lambda_2)(I + \tau\Lambda_1)(\varphi^{k+1} - \varphi^k) = -2\tau(\Lambda\varphi^k - f). \quad (9)$$

Let us note that in the right-hand side of equation (9) the residual of the iterative process is isolated. The general principles of using splitting schemes for solving the kinetic equation are considered in work (^{3}). In that same work the question of imposing boundary conditions for a completely split scheme is discussed in detail. The results formulated in the cited work concerning boundary conditions remain valid for the algorithm developed above.

The implementation scheme for the iterative process (9) will be as follows. Let us introduce auxiliary functions φ_1 and φ_2. Then we shall have

$$(I+\tau\Lambda_2)\varphi_1^{k+1} = -2\tau F^k, \qquad (I+\tau\Lambda_1)\varphi_2^{k+1} = \varphi_1^{k+1}, \qquad \varphi^{k+1} = \varphi^k + \varphi_2^{k+1}, \tag{10}$$

where

$$F^k = \Lambda\varphi^k - f. \tag{11}$$

To implement the system of equations (10), it is necessary to formulate an algorithm for solving the second of the equations of system (10). For this purpose it is convenient to introduce the notation $\psi = \varphi_2^{k+1}$, $g = \varphi_1^{k+1}$. Then we arrive at the following problem

$$(I+\tau\Lambda_1)\psi = g, \qquad \psi|_\Gamma = 0, \quad \text{if } \vec{\Omega}\mathbf{n} < 0. \tag{12}$$

Let us write problem (12) in expanded form and transform it somewhat:

$$\Omega_x\psi_x + \Omega_y\psi_y + \Omega_z\psi_z + \frac{1}{\tau}\psi = \frac{1}{\tau}g, \qquad \psi|_\Gamma = 0 \text{ for } \vec{\Omega}\mathbf{n} < 0. \tag{13}$$

Here the following notation has been used:

$$\vec{\Omega} = \Omega_x\mathbf{i} + \Omega_y\mathbf{j} + \Omega_z\mathbf{k}, \qquad \psi_s = \partial\psi/\partial s. \tag{14}$$

Introduce the formal parameter ξ into consideration and replace problem (13) by the following:

$$\Psi_\xi + \Omega_x\Psi_x + \Omega_y\Psi_y + \Omega_z\Psi_z = 0, \qquad \Psi|_{\xi=0} = \frac{1}{\tau}g, \qquad \Psi|_\Gamma = 0. \tag{15}$$

It is not difficult to verify that the solution of problem (14) is found in the form

$$\psi = \int_0^\infty \Psi e^{-\xi/\tau}\, d\xi. \tag{16}$$

Equation (15) admits splitting into one-dimensional problems (5). For this purpose we divide the interval $0 \leq \xi \leq \xi_0$ into partial intervals of width $\Delta\xi = \xi_{p+1} - \xi_p$. Then we shall have

$$\Psi_{1\xi} + \Omega_x \Psi_{1x} = 0, \quad \Psi_1 = \Psi^p, \qquad \Psi_{2\xi} + \Omega_y \Psi_{2y} = 0, \quad \Psi_2^p = \Psi_1^{p+1},$$
$$\Psi_{3\xi} + \Omega_z \Psi_{3z} = 0, \qquad \Psi_3^p = \Psi_2^{p+1}, \tag{17}$$

where the solution of the problem Ψ^{p+1} is found from the relation $\Psi^p = \Psi_3^{p+1}$. The resulting system of differential equations in partial derivatives makes it possible to find the exact solution of problem (15) on the system of points ξ_k. The extension of the required solution to the whole domain $0 \leq \xi \leq \xi_0$ is carried out by means of suitable quadrature formulas. Thus, we have arrived at the differential formulation of the problem of complete splitting of the kinetic equation.

Now we shall prove the convergence of the obtained splitting scheme (9) to the generalized solution of equation (8). Introduce the notation

$$T_\tau = (I + \tau\Lambda_1)^{-1}(I + \tau\Lambda_2)^{-1}(I - \tau\Lambda_2)(I - \tau\Lambda_1),$$
$$F = 2\tau(I + \tau\Lambda_1)^{-1}(I + \tau\Lambda_2)^{-1} f. \tag{18}$$

Then equation (9) can be reduced to the form

$$\varphi^{k+1} = T_\tau \varphi^k + F. \tag{19}$$

Let us give some properties of the operators Λ_1, Λ_2 in the spaces $\mathcal{L}_2(S \times G)$. The operator Λ_2 is bounded and self-adjoint in $\mathcal{L}_2$, and moreover transforms any function from $\mathcal{L}_2$ again into a function from $\mathcal{L}_2$.

Lemma 1. *The operator Λ_2 for $\sigma_c > 0$ is positive definite.*

Proof. Let $\varphi \in \mathcal{L}_2$. Then we have

$$(\Lambda_2\varphi, \varphi) = \int_{S\times G} \sigma(\mathbf{r})\varphi^2 \, d\mathbf{r} \, d\vec{\Omega} - \frac{1}{4\pi} \int_G \sigma_s(\mathbf{r}) \left(\int_S \varphi \, d\vec{\Omega} \right)^2 d\mathbf{r}. \tag{20}$$

Applying the Cauchy-Bunyakovsky-Schwarz inequality to the inner integral, we arrive at the assertion of the lemma, i.e.

$$(\Lambda_2\varphi, \varphi) \geq \inf_{r\in G} \operatorname{vrai} \sigma_c(\mathbf{r}) \|\varphi\|^2. \tag{21}$$

Lemma 2. *The operator $(I + \tau\Lambda_2)^{-1}$ exists, is bounded, and maps every function from $\mathcal{L}_2$ into $\mathcal{L}_2$; moreover the estimate holds*

$$\left\|(I+\tau\Lambda_2)^{-1}\right\| \le \sup_{r\in G}\operatorname{vrai}\frac{1}{1+\tau\sigma_c(\mathbf{r})}. \tag{22}$$

Proof. From Lemma 1 it follows that the operator $(I+\tau\Lambda_2)^{-1}$ exists for $\tau > 0$. We prove the second part of the lemma. Let $F \in \mathcal{L}_2$. Then, solving the equation $(I+\tau\Lambda_2)\varphi = F$, we have

$$\varphi = (I+\tau\Lambda_2)^{-1}F = \frac{\tau\sigma_s}{4\pi(1+\tau\sigma)(1+\tau\sigma_c)}\int_S F\,d\vec{\Omega} + \frac{1}{1+\tau\sigma}F. \tag{23}$$

Hence we arrive at estimate (22).

Lemma 3.

$$\left\|(I+\tau\Lambda_2)^{-1}(I-\tau\Lambda_2)\right\| \le \gamma < 1 \quad \text{for } \sigma_c > 0. \tag{24}$$

The assertion of the lemma follows from Lemma 1.

We next establish a number of properties of the operator Λ_1. The operator is non-self-adjoint.

Lemma 4. *For all $\tau > 0$ the operator $I+\tau\Lambda_1$ is positive definite.*

Proof. Let $\varphi \in D$. Then, taking into account condition 3) and the results of [1], we obtain

$$((I+\tau\Lambda_1)\varphi,\varphi) = \frac{1}{2}\int_{S\times G}\varphi^2(r'+\eta_1\vec{\Omega},\vec{\Omega})\,dr'\,d\vec{\Omega} + \frac{1}{\tau}\|\varphi\|^2.$$

Corollary. The operator Λ_1 is positive.

Lemma 5. The operator $(I+\tau\Lambda_1)^{-1}$ exists and is bounded from $\mathcal{L}_2$ into D, and

$$\|(I+\tau\Lambda_2)^{-1}\| \le ((1-e^{-(1/\tau)d}))\|F\|.$$

Here d is the diameter of the domain G.

The lemma is easy to prove on the basis of the results of [1].

Lemma 6. For all $\tau > 0$,

$$\left\|(I+\tau\Lambda_1)^{-1}(I-\tau\Lambda_1)\right\| \le 1.$$

The assertion of the lemma is obvious by virtue of the corollary to Lemma 4.

For what follows it is useful to introduce norms in the linear set D, setting

$$\|\varphi\|_{D_2} = \|(I+\tau\Lambda_1)\varphi\|, \qquad \varphi \in D.$$

We note the inequality

$$\|\varphi\| \leq (1 - e^{-(1/\tau)d})\|\varphi\|_{D_2}, \tag{25}$$

which holds for all φ from D by virtue of Lemma 5. Hence convergence in D_2 implies convergence in $\mathcal{L}_2$. The space D_2 is complete.

Theorem 1. For any initial approximation $\varphi^0 \in D$, the process of successive approximations (19) for equations (8) converges in $\mathcal{L}_2$ to the unique solution φ^* of equation (8) for any $\tau > 0$.

Proof. It is obvious that T_τ is linear and maps the set D into itself. In order to establish convergence of the process (19), it is sufficient to show

$$\|T_\tau\| \leq \chi < 1. \tag{26}$$

Let $\varphi \in D$. Then

$$\|T_\tau \varphi\|_{D_2} = \|(I + \tau\Lambda_1)T_\tau\varphi\| = \|P_\tau(I + \tau\Lambda_1)\varphi\| \leq \|P_\tau\| \, \|\varphi\|_{D_2},$$

where

$$P_\tau = (I + \tau\Lambda_2)^{-1}(I - \tau\Lambda_2)(I + \tau\Lambda_1)^{-1}(I - \tau\Lambda_1).$$

In this case we obtain

$$\|T_\tau\|_{D_2} \leq \|P_\tau\|. \tag{27}$$

Next we have

$$\|P_\tau\| \leq \|(I + \tau\Lambda_2)^{-1}(I - \tau\Lambda_2)\| \, \|(I + \tau\Lambda_1)^{-1}(I - \tau\Lambda_1)\|. \tag{28}$$

Hence, on the basis of Lemmas 4 and 6, we arrive at the inequality $\|P_\tau\| \leq \chi < 1$, and, consequently, on the basis of inequalities (24), (27) we obtain (26).

From the convergence of the process of successive approximations (19) it follows that

$$\lim_{k\to\infty} \varphi^k = \varphi^*.$$

To verify this, using the linearity of the operator T_τ, it is sufficient to pass to the limit in equation (19) as $k \to \infty$.

In conclusion we note the following. We proved Theorem 1 for $\sigma_s < \sigma$, i.e., the case of a purely scattering medium, when $\sigma_s = \sigma$, was excluded from consideration. However, by carrying out more precise estimates, one can prove Theorem 1 for the case $\sigma = \sigma_s$.

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