# ON A GENERALIZATION OF THE CONCEPT OF A LIE ALGEBRA

MATHEMATICS

1965

SovietRxiv

## Abstract

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UDC 512.934

*MATHEMATICS*

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# ON A GENERALIZATION OF THE CONCEPT OF A LIE ALGEBRA

*(Presented by Academician P. S. Novikov on 10 IV 1965)*

In the present note we consider a class of algebras generalizing the class of Lie algebras, and also containing algebras that are naturally equivalent to modules over Lie algebras. The ground field is everywhere assumed to have characteristic zero. Recall that a distributive algebra is a vector space $\mathcal{D}$ over a field $P$, endowed with an algebraic binary operation $a * b$, for which both distributive laws are satisfied, as well as the laws of unitarity, associativity, commutativity, and distributivity for multiplication by elements of $P$. Distributive algebras form a natural domain in which congruences defining homomorphisms are given in terms of two-sided ideals (see ([1]).

**Definition 1.** A distributive algebra $\mathcal{D}$ is called a **left $D$-algebra** if, for any $a, b, c \in \mathcal{D}$, the identity

$$a * (b * c) = (a * b) * c + b * (a * c). \tag{1}$$

is satisfied. Correspondingly, $\mathcal{D}$ is called a **right $D$-algebra** if

$$(a * b) * c = a * (b * c) + (a * c) * b, \tag{2}$$

and a **two-sided $D$-algebra** if both identities are satisfied. Equality (1) is called the **left differential identity**, and (2) the **right differential identity**.

A fixed algebra of one of these three types will be called a $D$-algebra. It may be noted that (1) and (2), generally speaking, are not consequences of one another. Indeed, in (1) the element $c$ does not change its place, while in (2) it does; correspondingly, in (2) the element $a$ does not change its place, while in (1) it does.

**Definition 2.** The **left center** of a distributive algebra $\mathcal{D}$ is the subspace

$$^*\mathcal{D} = \{x \mid x * a = 0 \text{ for all } a \in \mathcal{D}\}.$$

The **right center** in $\mathcal{D}$ is the subspace

$$\mathcal{D}^* = \{x \mid a * x = 0 \text{ for all } a \in \mathcal{D}\}.$$

Here $\mathcal{D}^*$ is a left ideal and ${}^*\mathcal{D}$ is a right ideal in $\mathcal{D}$.

**Theorem 1.** *If $\mathcal{D}$ is a left D-algebra, then ${}^*\mathcal{D}$ is a two-sided ideal in $\mathcal{D}$, and $\mathcal{D}/{}^*\mathcal{D}$ is a Lie algebra.*

**Proof.** Interchanging $a$ and $b$ in (1) and adding the resulting equality to (1), we find that, for all $a, b \in \mathcal{D}$,

$$a * b + b * a \in {}^*\mathcal{D}. \tag{3}$$

Since ${}^*\mathcal{D}$ is a right ideal, it follows from this that ${}^*\mathcal{D}$ is also a left ideal, and in $\mathcal{D}/{}^*\mathcal{D}$ $\bar{a} * \bar{b} + \bar{b} * \bar{a} = \bar{0}$; consequently, $\mathcal{D}/{}^*\mathcal{D}$ is an anticommutative $D$-algebra, i.e., a Lie algebra.

As a consequence we obtain the theorem: if $\mathcal{D}$ is a left $D$-algebra and ${}^*\mathcal{D} = 0$, then $\mathcal{D}$ is a Lie algebra. Let us note that the minimal ideal in $\mathcal{D}$,

factor algebra of which is a Lie algebra, is the two-sided ideal $\mathcal{D}_0$ generated by all elements of the form $a * b + b * a;\ a, b \in \mathcal{D}$.

**Proposition 1.** $\mathcal{D}$ is a left $D$-algebra, $\mathcal{F} \subseteq \mathcal{D}$ is a two-sided ideal such that $\mathcal{D}/\mathcal{F}$ is a semisimple Lie algebra. Then ${}^*\mathcal{D} \subseteq \mathcal{F}$.

**Proof.** The center of a semisimple algebra is zero; in $\mathcal{D}/\mathcal{F}$ this gives: from $\bar{x} * \bar{a} = \bar{0}$ for all $\bar{a} \in \mathcal{D}/\mathcal{F}$ it follows that $\bar{x} = \bar{0}$, i.e., from $x * a \in \mathcal{F}$ for all $a \in \mathcal{D}$ it follows that $x \in \mathcal{F}$. The theorem follows from this observation, since $0 \in \mathcal{F}$.

**Example 1.** Suppose that the left $D$-algebra $\mathcal{D}$ contains, as a subalgebra, a Lie algebra $\mathcal{L} \approx \mathcal{D}/\mathcal{F}$, and moreover $\mathcal{D}_0 \subseteq \mathcal{F} \subseteq {}^*\mathcal{D}$. From (1), rewritten in the form

$$(a * b) * x = a * (b * x) - b * (a * x), \tag{4}$$

we note that $\mathcal{F}$, with respect to left multiplication by elements of $\mathcal{L}$, i.e., with respect to the left adjoint representation, is a left $\mathcal{L}$-module. Conversely, every left $\mathcal{L}$-module $\mathcal{F}$ over the Lie algebra $\mathcal{L}$ can be turned into a $D$-algebra by putting $\mathcal{D} = \mathcal{F} \dotplus \mathcal{L}$ and defining the multiplication operation in $\mathcal{D}$ by the formulas: $a * b = [ab]$, $a * x = ax$, $x * a = 0$, $x * y = 0$, where $a, b \in \mathcal{L};\ x, y \in \mathcal{F}$. The correctness of the definition follows from (4).

**Proposition 2.** For a semisimple Lie ideal $\mathcal{L}$ contained in a finite-dimensional $D$-algebra $\mathcal{D}$, there exists an ideal complementary to it.

**Proof.** Noting that $\mathcal{L} \cap {}^*\mathcal{D}$ is an ideal in $\mathcal{L}$ contained in the center of $\mathcal{L}$, by virtue of the semisimplicity of $\mathcal{L}$ we find: $\mathcal{L} \cap {}^*\mathcal{D} = 0$. Consequently, the natural homomorphism $i : \mathcal{D} \to \mathcal{D}/{}^*\mathcal{D}$ is a monomorphism on $\mathcal{L}$. Put $\operatorname{Im} \mathcal{L} = \bar{\mathcal{L}}$. Applying Levi' s theorem and the Malcev-Harish-Chandra lemma, we obtain that for every semisimple Lie ideal in a Lie algebra there exists an ideal complementary to it (see ([2])). In the present case put $\mathcal{D}/{}^*\mathcal{D} = \bar{\mathcal{L}} \oplus \bar{\mathcal{F}}$. It is now clear that $i^{-1}(\bar{\mathcal{F}})$ is an ideal in $\mathcal{D}$ complementary to $\mathcal{L}$.

**Example 2.** **$\mathcal{L}$-Lie algebra.** We shall call two $\mathcal{L}$-modules $\mathfrak{M}_1$ and $\mathfrak{M}_2$ $D$-connected if the following conditions are satisfied: 1) $\mathfrak{M}_1$ is a right $\mathcal{L}$-module, $\mathfrak{M}_2$ is a left $\mathcal{L}$-module; 2) $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are defined on one vector space $\mathfrak{M}$; 3) for all $a, b \in \mathcal{L},\ x \in \mathfrak{M}$ the equalities hold

$$(ax)b + b(ax) = 0, \quad (ax)b + (xa)b = 0; \quad b(ax) + b(xa) = 0.$$

For $D$-connected $\mathcal{L}$-modules, on the vector space $\mathfrak{M} \dotplus \mathcal{L}$ the structure of a two-sided $D$-algebra is defined in the following way: put, for $a, b \in \mathcal{L}$; $x, y \in \mathfrak{M}$, $a*b = [ab]$, $a*x = ax$; $x*a = xa$; $x*y = 0$, and require that this multiplication law be distributive. Concerning this example we make two remarks.

I. If $\mathfrak{M}_1$ is an $\mathcal{L}$-module anti-isomorphic to the $\mathcal{L}$-module $\mathfrak{M}_2$, then the constructed $D$-algebra coincides with the split extension of $\mathcal{L}$ by means of $\mathfrak{M}$, where $\mathfrak{M}$ is endowed with the structure of an abelian Lie algebra.

II. From the relations defining $D$-connected $\mathcal{L}$-modules there follows the equality $[ab]x = x[ba]$, so that this notion coincides with the notion of anti-isomorphic $\mathcal{L}$-modules, for example, if $\mathcal{L}$ is a semisimple Lie algebra.

The class of $D$-algebras is a primitive class of universal algebras; in particular, for a set of a given cardinality there exists, up to isomorphism, a unique $D$-algebra with this set of generators. In the case of Lie algebras, Witt' s theorem holds on the connection between the free associative algebra and the free Lie algebra with the same set of generators. As will now be shown, this connection does not hold for free $D$-algebras. Denote by $\mathfrak{A}$ an associative algebra.

**Definition 3.** A distributive algebra $\mathcal{D}$ is called **associated with $\mathfrak{A}$ under a mapping** $f : \mathfrak{A} \times \mathfrak{A} \to \mathfrak{A}$, if the following conditions are satisfied: 1) $\mathcal{D}$ coincides with $\mathfrak{A}$ as a set and as a vector space; 2) $a * b = f(a, b)$ for all $a, b \in \mathcal{D}$. $\mathcal{D}$ is polynomially associated with $\mathfrak{A}$ if $f$ is a polynomial over $\mathfrak{A}$.

**Theorem 2.** *If $\mathfrak{A}$ is a free algebra, then all $D$-algebras polynomially associated with $\mathfrak{A}$ are Lie algebras, and the multiplication formula in each of them has the form*

$$[ab] = \mu(ab - ba),$$

*where $\mu \in P$ is an arbitrary fixed element.*

**Proof.** If $f(x, y)$ is a polynomial over a free algebra, then from $f(a, b) = 0$ for all $b \in \mathfrak{A}$ it follows that either $f \equiv 0$, or $a = 0$. The first case is trivial; in the second we obtain that ${}^*\mathfrak{D} = 0$, and, by Theorem 1, $\mathfrak{D}$ is a $D$-algebra. One can verify that in this case the multiplication formula in $\mathfrak{D}$ must have the indicated form.

The $D$-algebras defined in Example 1 can be connected with the theory of representations of Lie groups. We shall take the ground field to be the field of real numbers. Let $G$ be a local Lie group for which $\mathcal{L}$ is the Lie algebra, and let

$$\varphi : G \times \mathfrak{F} \to \mathfrak{F}$$

be a representation of $G$ defined by the $\mathcal{L}$-module $\mathfrak{F}$. The representation $\varphi$ defines a (local) Lie group, which we denote by $E(G,\mathfrak{F})$, in the following way. Put $E(G,\mathfrak{F}) = G \times \mathfrak{F}$ as a manifold and define multiplication in $E(G,\mathfrak{F})$ by the formula

$$(g_1, x_1)(g_2, x_2) = (g_1 g_2,\, g_1 x_2 + x_1); \qquad g_1, g_2 \in G, \quad x_1, x_2 \in \mathfrak{F}.$$

Next, define the operation of $*$-commutation in $E(G,\mathfrak{F})$, generalizing the operation of commutation in a group:

$$(g_1, x_1) * (g_2, x_2) = ([g_1 g_2], 0)(1, g_1 x_2 - x_2),$$

where $[g_1 g_2]$ denotes the commutator of the elements of $G$. Consider now the tangent space $\mathfrak{D}$ to $E(G,\mathfrak{F})$ at the identity. Addition of vectors and multiplication of a vector by a number in $\mathfrak{D}$ are related in the usual way to multiplication and to a change of parametrization in the set $\Psi$ of all twice differentiable paths in $E(G,\mathfrak{F})$ passing through the identity and parametrized so that

$$a(t)\big|_0 \equiv (g(t), x(t))\big|_0 = (1, 0)$$

(see (3)).

Denote by $\partial a(t)$ the vector tangent to $a(t) \in \Psi$ at the identity. Let $a(t), b(t) \in \Psi$; put $(a * b)(t)$ to be the path obtained from the path $a(t) * b(t)$ by replacing the parameter $t$ by $\sqrt{t}$. Introducing into $\mathfrak{D}$ the operation of multiplication $*$ by the formula

$$\partial a(t) * \partial b(t) = \partial (a * b)(t),$$

we obtain a distributive algebra, since the embeddings

$$i_1 : G \to E(G,\mathfrak{F}) \quad \text{and} \quad i_2 : \mathfrak{F} \to E(G,\mathfrak{F})$$

are homomorphisms and

$$(g, x) = (g, 0)(1, x).$$

Now homomorphisms

$$\mathcal{L} \to \mathfrak{D} \quad \text{and} \quad \mathfrak{F} \to \mathfrak{D}$$

are naturally defined, where $\mathfrak{F}$ is endowed with the structure of an Abelian Lie algebra. In this case

$$\mathfrak{D} = \mathcal{L} + \mathfrak{F}$$

as a vector space, and the multiplication in $\mathfrak{D}$ coincides with that defined in Example 1.

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Received
9 IV 1965

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*Note: Figure translations are in progress. See original paper for figures.*