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Abstract

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ON CONDITIONS FOR THE UNITARY EQUIVALENCE OF COMMUTATIVE SYMMETRIC ALGEBRAS IN THE SPACE Π_k

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1. In the author' s preceding article ([1]), a description was given of commutative symmetric algebras (c.s.a.) in the Pontryagin space Π_k. In the present article necessary and sufficient conditions are given for the equivalence of two c.s.a.' s realized in accordance with Theorem 1 in ([1]); the terminology and notation of article ([1]) are retained. For the case $k = 1$, equivalence conditions were indicated earlier by the author in ([2,3]).

Let $R, \widetilde{R}$ be equivalent c.s.a.' s in the spaces $\Pi_k, \widetilde{\Pi}_k$, with only real eigenfunctionals (e.f.), realized by means of the decompositions

$$\Pi_k = (\mathfrak{N} \dotplus \mathfrak{N}') \oplus \mathfrak{H} \oplus \Pi, \tag{1}$$

$$\widetilde{\Pi}_k = (\widetilde{\mathfrak{N}} \dotplus \widetilde{\mathfrak{N}}') \oplus \widetilde{\mathfrak{H}} \oplus \widetilde{\Pi} \tag{1'}$$

of the algebras $R_1, R_2, \widetilde{R}_1, \widetilde{R}_2$ in $\mathfrak{H}, \Pi, \widetilde{\mathfrak{H}}, \widetilde{\Pi}$, respectively, of biorthogonal bases $\{x_{jl}\}$ in $\mathfrak{N}$, $\{y_{jl}\}$ in $\mathfrak{N}'$; $\{\tilde{x}_{jl}\}$ in $\widetilde{\mathfrak{N}}$, $\{\tilde{y}_{jl}\}$ in $\widetilde{\mathfrak{N}}$, and defining many-valued mappings $\Xi, \widetilde{\Xi}$. Let U be an operator which maps Π_k isometrically onto $\widetilde{\Pi}_k$, such that the operators $\widetilde{A} = UAU^{-1}$, $A \in R$, form precisely the algebra $\widetilde{R}$. It is not hard to verify that then:

I. If $\lambda_1(A), \dots, \lambda_p(A)$ are all the distinct e.f.' s of the algebra R, then $\tilde{\lambda}_1(\widetilde{A}), \dots, \tilde{\lambda}_p(\widetilde{A})$, where $\tilde{\lambda}_j(\widetilde{A}) = \lambda_j(A)$ for $\widetilde{A} = UAU^{-1}$, are all the distinct e.f.' s of the algebra $\widetilde{R}$.

II. If $\mathfrak{P}, \mathfrak{M}, \mathfrak{N}; \widetilde{\mathfrak{P}}, \widetilde{\mathfrak{M}}, \widetilde{\mathfrak{N}}$ are the principal, basic, and basic null spaces of the algebras R and $\widetilde{R}$, then

$$\widetilde{\mathfrak{P}} = U\mathfrak{P}, \qquad \widetilde{\mathfrak{M}} = U\mathfrak{M}, \qquad \widetilde{\mathfrak{N}} = U\mathfrak{N}.$$

III. If $\mathfrak{P}$ is a nonnegative k-dimensional subspace in Π_k, invariant with respect to all $A \in R$, and $\mathfrak{P} = \mathfrak{P}_1 \oplus \dots \oplus \mathfrak{P}_p$ is its decomposition into root lineals in $\mathfrak{P}$, corresponding to $\lambda_1, \dots, \lambda_p$, then

$$\widetilde{\mathfrak{P}} = \widetilde{\mathfrak{P}}_1 \oplus \dots \oplus \widetilde{\mathfrak{P}}_p,$$

where $\widetilde{\mathfrak{P}} = U\mathfrak{P}$, $\widetilde{\mathfrak{P}}_j = U\mathfrak{P}_j$, is the decomposition of $\widetilde{\mathfrak{P}}$ into root lineals in $\widetilde{\mathfrak{P}}$, corresponding to $\tilde{\lambda}_1, \dots, \tilde{\lambda}_p$.

An analogous assertion is valid if $\mathfrak{P}, \mathfrak{P}_j$ are replaced by the subspaces $\mathfrak{N}, \mathfrak{N}_j$.

Put

$$x'_{jl} = U^{-1}\tilde{x}_{jl}, \qquad y'_{jl} = U^{-1}\tilde{y}_{jl}, \qquad U^{-1}\widetilde{\mathfrak{N}} = \mathfrak{N}'. \tag{2}$$

Then $\{x'_{jl}\}$ is a basis in $\mathfrak{N}'$, $\{y'_{jl}\}$ is a basis in $\widehat{\mathfrak{N}}'$, biorthogonal to $\{x'_{jl}\}$; $\mathfrak{N}, \widehat{\mathfrak{N}}'$ are skew-related. Moreover, $\{x_{jl}\}$, as well as $\{x'_{jl}\}$ for fixed j, form a basis in $\mathfrak{N}_j$; consequently,

$$x'_{jl} = \sum_{s=1}^{r_j} a_{jls} x_{js}, \tag{3}$$

where $a_j = \|a_{jls}\|$, $l, s = 1, \dots, j$, is a nonsingular matrix. Further putting $U^{-1}\mathfrak{H} = \mathfrak{H}'$, $U^{-1}\Pi = \Pi'$, we have

$$\mathfrak{H}' = \mathfrak{M} \cap \mathfrak{H}'^{\perp}, \quad \Pi' = \mathfrak{L} \cap \mathfrak{H}'^{\perp}, \quad \mathfrak{M} = \mathfrak{R} \oplus \mathfrak{H}', \quad \mathfrak{L} = \mathfrak{M} \oplus \Pi'; \tag{4}$$

$$\Pi_k = (\mathfrak{R} \dotplus \mathfrak{R}') \oplus \mathfrak{H}' \oplus \Pi'. \tag{5}$$

Moreover,

$$\Pi_k = (\mathfrak{R} \dotplus \mathfrak{R}') \oplus \mathfrak{H} \oplus \Pi. \tag{6}$$

By virtue of the third relation in (4), every element $h \in \mathfrak{H}$ can be represented in the form

$$h = \sum_{j=1}^{q} \sum_{l=1}^{r_j} (h, y'_{jl}) x'_{jl} + h', \quad h' \in \mathfrak{H}'. \tag{7}$$

On the other hand, applying (6) to y'_{jl} and taking (3) into account, we obtain

$$y'_{jl} = \sum_{\mu=1}^{q} \sum_{\nu=1}^{r_\mu} \gamma_{jl\mu\nu} x_{\mu\nu} + \sum_{\nu=1}^{r_j} \bar{b}_{j\nu l} y_{j\nu} + h^0_{jl} + \pi^0_{jl}, \tag{8}$$

where $\gamma_{jl\mu\nu} = (y'_{jl}, y_{\mu\nu})$, $h^0_{jl} \in \mathfrak{H}$, $\pi^0_{jl} \in \Pi$, $b_j = \|b_{j\nu l}\|$, $\nu, l = 1, \dots, r_j$, is the matrix inverse to a_j: $b_j = a_j^{-1}$. From (8) we conclude that $(h, y'_{jl}) = (h, h^0_{jl})$, and therefore (7) is rewritten in the form

$$h = \sum_{j=1}^{q} \sum_{l=1}^{r_j} (h, h^0_{jl}) x'_{jl} + h', \quad h' \in \mathfrak{H}'. \tag{9}$$

Define the operator W_1 from $\mathfrak{H}$ into $\mathfrak{H}'$ by putting

$$W_1 h = h' = -\sum_{j=1}^{q} \sum_{l=1}^{r_j} (h, h^0_{jl}) x'_{jl} + h. \tag{10}$$

IV. *The operator W_1 maps $\mathfrak{H}$ isometrically onto $\mathfrak{H}'$, and the inverse operator is given by the formula*

$$W_1^{-1} h' = \sum_{j=1}^{q} \sum_{l=1}^{r_j} (h', h^0_{jl}) x_{jl} + h'. \tag{11}$$

An analogous assertion is valid for the operator W_2 from Π onto Π', defined by the formula

$$W_2 \pi = \pi' = -\sum_{j=1}^{q} \sum_{l=1}^{r_j} (\pi, \pi^0_{jl}) x'_{jl} + \pi. \tag{12}$$

Therefore the formulas $V_1 = UW_1$, $V_2 = UW_2$ define isometric operators from $\mathfrak{H}$ and Π onto $\mathfrak{H}$ and Π, respectively.

Let now the operators $A \in R$ and $\widetilde{A} = U^{-1} A U \in \widetilde{R}$ be given, with the aid of the systems $\xi = \{\lambda_{jls}, \alpha_{jl\mu s}, h_{jl}, \pi_{jl}, A_1, A_2\} \in \Xi$, $\tilde{\xi} = \{\tilde{\lambda}_{jls}, \tilde{\alpha}_{jl\mu s}, \tilde{h}_{jl}, \widetilde{A}_1, A_2\} \in \Xi$, by the formulas

$$Ax_{jl} = \sum_{s=1}^{l} \lambda_{jls} x_{js}, \quad Ah = \sum_{j=1}^{q} \sum_{l=1}^{r_j} (h, h_{jl}) x_{jl} + A_1 h,$$

$$A\pi = \sum_{j=1}^{q} \sum_{l=1}^{r_j} (\pi, \pi_{jl}) x_{jl} + A_2 \pi, \tag{13}$$

$$Ay_{jl} = \sum_{\mu=1}^{q} \sum_{s=1}^{r_\mu} \alpha^*_{jl\mu s} x_{\mu s} + \sum_{\mu=l}^{r_j} \lambda^*_{j\mu l} y_{j\mu} + h^*_{jl} + \pi^*_{jl},$$

$h, h_{jl}, h_{jl}^* \in \mathfrak{H};\ \pi, \pi_{jl}, \pi_{jl}^* \in \Pi$ and

$$\widetilde{A}\tilde{x}_{jl} = \sum_{s=1}^{b} \tilde{\lambda}_{jls}\tilde{x}_{js}, \quad \widetilde{A}\tilde{h} = \sum_{j=1}^{q}\sum_{l=1}^{r_j}(\tilde{h}, \tilde{h}_{jl})\tilde{x}_{jl} + \widetilde{A}_1\tilde{h}, \quad \widetilde{A}\tilde{\pi} = \sum_{j=1}^{q}\sum_{l=1}^{r_j}(\tilde{\pi}, \tilde{\pi}_{jl})\tilde{x}_{jl} +$$

$$+\widetilde{A}_2\tilde{\pi}, \quad \widetilde{A}y_{jl} = \sum_{\mu=1}^{q}\sum_{s=1}^{r_\mu}\tilde{\alpha}_{jl\mu s}^* x_{\mu s} + \sum_{\mu=1}^{r_j}\bar{\tilde{\lambda}}_{j\mu l}y_{j\mu} + \tilde{h}_{jl}^* + \tilde{\pi}_{jl}^*, \tag{14}$$

$\tilde{h}, \tilde{h}_{jl}, \tilde{h}_{jl}^* \in \widetilde{\mathfrak{H}};\ \tilde{\pi}, \tilde{\pi}_{jl}, \tilde{\pi}_{jl}^* \in \widetilde{\Pi}$ (see (3.8)–3.10) in ([1]). Putting in (14) $\widetilde{A} = UAU^{-1}$, $h' = U^{-1}\tilde{h}$, $\pi' = U^{-1}\tilde{\pi}$, $h = W_1^{-1}h'$, $\pi = W_2^{-1}\pi'$ and using (2), (3), (8), (10)–(13), we arrive at the following result:

Theorem 1. Let $R, \widetilde{R}$ be commutative symmetric algebras with only real eigenfunctions in the spaces $\Pi_k, \widetilde{\Pi}_k$, given by means of the decompositions (1), (1′), the bases $\{x_{jl}\}$, $\{y_{jl}\}$, $\{\tilde{x}_{jl}\}$, $\{\tilde{y}_{jl}\}$, $l = 1, \dots, r_j;\ j = 1, \dots, q$, in $\mathfrak{R}, \mathfrak{R}', \widetilde{\mathfrak{R}}, \widetilde{\mathfrak{R}}'$, the algebras $R_1, \widetilde{R}_1, R_2, \widetilde{R}_2$ in $\mathfrak{H}, \widetilde{\mathfrak{H}}, \Pi, \widetilde{\Pi}$, respectively, and the determining manifolds $\Xi, \widetilde{\Xi}$ associated with them. The algebras $R, \widetilde{R}$ are equivalent if and only if there exist: a) nonsingular matrices $a_j = \|a_{jls}\|$, $l, s = 1, \dots, r_j;\ j = 1, \dots, q$; b) numbers $\gamma_{jl\mu\nu}$, $l = 1, \dots, r_j;\ \nu = 1, \dots, r_\mu;\ j, \mu = 1, \dots, q$; c) elements $h_{jl}^0 \in \mathfrak{H}$, $\pi_{jl}^0 \in \Pi$, $l = 1, \dots, r_j;\ j = 1, \dots, q$; d) isometric mappings V_1, V_2 of the spaces $\mathfrak{H}, \Pi$ onto $\widetilde{\mathfrak{H}}, \widetilde{\Pi}$ such that:

$$1)\quad \sum_{\nu=1}^{r_\mu}\gamma_{jl\mu\nu}b_{\mu\nu s} + \sum_{\nu=1}^{r_j}\bar{b}_{j\nu l}\bar{\gamma}_{\mu s j\nu} + (h_{jl}^0, h_{\mu s}^0) + (\pi_{jl}^0, \pi_{\mu s}^0) = 0,$$

where

$$b_j = \|b_{jll'}\| = a_j^{-1}, \quad l, l' = 1, \dots, r_j;\ s = 1, \dots, r_\mu;\ j, \mu = 1, \dots, q;$$

2) the formulas

$$\tilde{\lambda}_{jls} = \sum_{\mu=1}^{r_j}\sum_{\nu=1}^{\mu}a_{jl\mu}\lambda_{j\mu\nu}b_{j\nu s}, \quad \widetilde{A}_1 = V_1A_1V_1^{-1}, \quad \widetilde{A}_2 = V_2A_2V_2^{-1}, \tag{15}$$

$$\tilde{h}_{js} = V_1\left(A_1^*h_{js}^0 - \sum_{l=s}^{r_j}\bar{\tilde{\lambda}}_{jls}h_{jl}^0 + \sum_{l=1}^{r_j}\bar{b}_{jls}h_{jl}\right), \tag{16}$$

$$\tilde{\pi}_{js} = V_2\left(A_2^*\pi_{js}^0 - \sum_{l=s}^{r_j}\bar{\tilde{\lambda}}_{jls}\pi_{jl}^0 + \sum_{l=1}^{r_j}\bar{b}_{jls}\pi_{jl}\right), \tag{17}$$

$$\tilde{\alpha}_{jl\mu\nu} = \sum_{s=1}^{q}\sum_{\tau=1}^{q} \bar{b}_{jsl}\alpha_{js\mu\tau}b_{\mu\tau\nu} + \sum_{\tau=1}^{q}\sum_{s=1}^{\tau}\gamma_{jl\mu\tau}\lambda_{\mu\tau s}b_{\mu s\nu} + \sum_{\tau=1}^{q}\sum_{s=1}^{\tau}\bar{b}_{jsl}\bar{\lambda}_{j\tau s}\bar{\gamma}_{\mu\nu j\tau}$$

$$+ \sum_{s=1}^{q}\bar{b}_{jsl}[(h^*_{js}, h^0_{\mu\nu}) + (\pi^*_{js}, \pi^0_{\mu\nu})] +$$

$$+ \sum_{\tau=1}^{q} b_{\mu\tau\nu}[(h^0_{jl}, h_{\mu\tau}) + (\pi^0_{jl}, \pi_{\mu\tau})] + (A_1 h^0_{jl}, h^0_{\mu\nu}) + (A_2\pi^0_{jl}, \pi^0_{\mu\nu}) \qquad (18)$$

carry out a one-to-one mapping of Ξ onto $\widetilde{\Xi}$.

Condition 1) means that the space $\mathfrak{R}'$, spanned by $\{y'_{jl}\}$, is zero.

2. Suppose now that $\Pi_k, \widetilde{\Pi}_k$ are separable, that the algebras $R, \widetilde{R}$ are separable with respect to the operator norm and are given in the form of canonical models described in Sec. 5 of ([1]). Let

$$\mathfrak{H} = \int_T \mathfrak{H}(t)\, d\sigma, \quad \widetilde{\mathfrak{H}} = \int_{\widetilde{T}} \widetilde{\mathfrak{H}}(\tilde{t})\, d\tilde{\sigma}$$

be the corresponding realizations of the spaces $\widetilde{\mathfrak{H}}, \mathfrak{H}$ in these canonical models. Applying to the second formula in (15) the lemma from Appendix IV in ([4]), the argument on p. 223 in ([4]), and then using formula (16), we obtain:

Theorem 2. Let $R, \widetilde{R}$ be two equivalent canonical models, specified by means of the spaces

$$\int_T \mathfrak{H}(t)\, d\sigma, \qquad \int_{\widetilde{T}} \widetilde{\mathfrak{H}}(\tilde{t})\, d\tilde{\sigma},$$

the algebras $R_1, \widetilde{R}_1, R_2, \widetilde{R}_2$, the vector-functions $\xi_{jl}(t), \tilde{\xi}_{jl}(\tilde{t})$, and the defining manifolds Ξ and $\widetilde{\Xi}$. Then there exist: a) a homeomorphism s of the space T onto $\widetilde{T}$, mapping one-to-one the set $\{t_j,\ j = 1, \dots, m\}$ of all singular points of the algebra R onto the set $\{\tilde{t}_j,\ j = 1, \dots, m\}$ of all singular points of the algebra $\widetilde{R}$; b) a σ-measurable operator function $V_1(t)$, defined σ-almost everywhere on T, whose value for σ-almost every $t \in T$ is an isometric operator $V_1(t)$ mapping $\mathfrak{H}(t)$ onto $\widetilde{\mathfrak{H}}(st)$; c) vector-functions $h^0_{j\nu}(t) \in \mathfrak{H},\ \nu = 1, \dots, r_j;\ j = 1, \dots, q$, such that: 1) the measures $\tilde{\sigma}(s\Delta)$ and $\sigma(\Delta)$, $\Delta \subset T$, are equivalent; 2) the operator V in (15) is given by the formula $V\{h(t)\} = \{\tilde{h}(\tilde{t})\}$, where $\tilde{h}(st) = \rho(t)V_1(t)h(t)$, $\rho(t) = d\tilde{\sigma}(st)/d\sigma(t)$; 3) σ-almost everywhere on T_j

$$\tilde{\xi}_{j\nu}(st) = \rho(t)V_1(t)\left[h^0_{j\nu}(t) + \sum_{l=1}^{r_j}\bar{b}_{jl\nu}\xi_{jl}(t)\right],$$

and if t_j is a singular point,

$$\tilde{k}_{j\nu} = V_{1j}\left(\sum_{l=1}^{r_j} \bar{b}_{jl\nu} k_{jl} - \sum_{l=\nu+1}^{r_j} \overline{\lambda}_{jl\nu} k_{jl}\right),$$

where V_{1j} is the restriction of the operator V_1 to the singular space K_j.

Conversely, if all these conditions are satisfied, then: α) the operator V_1 maps $\mathfrak{H}$ onto $\tilde{\mathfrak{H}}$ in such a way that $\overline{R}_1$ is mapped onto $\widetilde{R}_1$; β) for the vector-functions

$$h_{jl}(t) = (A(t) - \lambda_j)\xi_{jl}(t) - \sum_{\mu=l+1}^{r_j} \lambda_{j\mu l}\xi_{j\mu}(t) \quad \text{for } t \neq t_j, \qquad h_{jl}(t_j) = k_{jl},$$

$$\tilde{h}_{jl}(\tilde{t}) = (\widetilde{A}(\tilde{t}) - \tilde{\lambda}_j)\tilde{\xi}_{jl}(\tilde{t}) - \sum_{\mu=l+1}^{r_j} \overline{\tilde{\lambda}}_{j\mu l}\tilde{\xi}_{j\mu}(\tilde{t}) \quad \text{for } \tilde{t} \neq \tilde{t}_j, \qquad \tilde{h}_{jl}(\tilde{t}_j) = \tilde{k}_{jl},$$

relation (16) is satisfied.

Corollary. Let R be a separable c.s.a. with only real c.f. in the separable space Π_k. If $\lambda_{m+1}, \dots, \lambda_q$ are all regular c.f. of the algebra R with eigenvectors on the principal null subspace $\mathfrak{N}$, then the skew-orthogonal subspace $\mathfrak{N}'$ can be chosen so that $h_{jl}(A) = 0$ for $j = m+1, \dots, q$.

Indeed, from the regularity of λ_j it easily follows that $\{\xi_{jl}(t)\} \in \mathfrak{H}$, and therefore one may set $h^0_{jl}(t) = -\xi_{jl}(t)$ for $j = m+1, \dots, q$; $h_{jl}(t) = 0$ for $j = 1, \dots, q$.

Put further

$$y'_{jl} = -\frac{1}{2}\sum_{\mu=1}^{q}\sum_{\nu=1}^{r_\mu}(h^0_{jl}, h^0_{\mu\nu})x_{\mu\nu} + y_{jl} + h^0_{jl}, \qquad l = 1, \dots, r_j; \quad j = 1, \dots, q,$$

and denote by $\mathfrak{N}'$ the subspace spanned by $\{y'_{jl}\}$. Then $\mathfrak{N}'$ is a null subspace skew-orthogonal to $\mathfrak{N}$, $\{y'_{jl}\}$ is a basis in $\mathfrak{N}'$, biorthogonal to $\{x_{jl}\}$; put $\mathfrak{H}' = \mathfrak{M} \cap \mathfrak{N}'^{\perp}$, $\Pi' = \Omega \cap \mathfrak{N}'^{\perp}$. Applying Theorems 1 and 2 to the algebras R and $\widetilde{R} = R$, realized by means of the decompositions $\Pi_k = (\mathfrak{N} \dotplus \mathfrak{N}') \oplus \mathfrak{H} \oplus \Pi$, $\Pi_k = (\mathfrak{N} \dotplus \mathfrak{N}') \oplus \mathfrak{H}' \oplus \Pi'$, the bases $\{x_{jl}\}$ in $\mathfrak{N}$, $\{y_{jl}\}, \{y'_{jl}\}$ in $\mathfrak{N}', \mathfrak{N}'$, respectively, we conclude that $\xi_{j\nu}(st) = \rho(t)V_1(t)[h^0_{j\nu}(t) + \xi_{j\nu}(t)] = 0$, and therefore $h_{jl}(A) = 0$ for $j = m+1, \dots, q$.

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Note: Figure translations are in progress. See original paper for figures.