ON THE QUESTION OF UNIQUENESS OF SOLUTIONS OF VOLTERRA INTEGRAL EQUATIONS WITH RETARDED ARGUMENT

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Abstract

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As is known, the equation $y' = f(t, y)$, $y(0) = y_0$, has a unique solution if

$$|f(t, y_1) - f(t, y_2)| \leq \omega(|y_1 - y_2|) \quad \text{and} \quad \int_0^{\varepsilon} \frac{dx}{\omega(x)} = \infty,$$

i.e., if $|\Delta f| = |f(t, y + \Delta y) - f(t, y)|$ is a slowly increasing function of Δy in a right neighborhood of zero.

A. D. Myshkis showed ([1]) that for the equation

$$y'(t) = f(t, y(t - g(t))) \tag{1}$$

one can weaken the requirement on the growth of $|\Delta f|$, if as $t \to 0$ the argument $t - g(t)$ tends to zero sufficiently rapidly. More precisely, if $t - g(t) \leq bt^{1/\alpha}$, $0 < \alpha < 1$, $b > 0$, then one may put $\omega(x) = x^{\alpha}$.

The present note is devoted to a further development and generalization of A. D. Myshkis' result; namely, it will be shown that whatever the order of growth of $|\Delta f|$, with a corresponding delay equation (1) has a unique solution.

Consider the system

$$x(t) = \int_a^t G(t, s, x(s - g(s)))\, ds + f(t), \qquad t \geq a, \tag{2}$$

$$x(t) = \psi(t), \qquad t \leq a.$$

Here

$$G(t, s, x) = \{G_1(t, s, x_1, \dots, x_n), \dots, G_n(t, s, x_1, \dots, x_n)\}$$

and

$$f(t) = \{f_1(t), \dots, f_n(t)\}$$

are vector-functions continuous for $a \le s \le t \le T$, $\|x\| < C$; $g(t)$ is continuous and nonnegative for $t \in [a, T)$; the vector-function

$$\psi(t) = \{\psi_1(t), \dots, \psi_n(t)\}$$

is continuous for $t \le a$ and $\psi(a) = f(a)$.

Denote by P the set of such points τ that $t - g(t) \le \tau$ for $t \in [\tau, \tau + \varepsilon]$ for some $\varepsilon > 0$.

Theorem 1. Let, for each point $t_0 \notin P$, there exist continuous increasing functions $\omega(\tau) \ge \tau$ and $\varphi(\tau)$ $(\omega(0) = \varphi(0) = 0)$, for $\tau \in [0, \varepsilon]$, such that:

1. $$\|G(t, s, x) - G(t, s, y)\| \le K\omega(\|x - y\|)$$
   $$(t_0 \le s \le t \le t_0 + \varepsilon,\ \|x\| < C,\ \|y\| < C,\ \|x - y\| \le \varepsilon).$$
2. $$t - g(t) \le t_0 + \omega_{-1}[\varphi(t - t_0)], \qquad t \in [t_0, t_0 + \varepsilon],$$
   where $\omega_{-1}(\tau)$ is the function inverse to $\omega(\tau)$.
3. $$\omega_{-1}[\varphi(\tau)] \ge \varphi[\omega_{-1}(\tau)], \qquad \varphi(\tau) < \tau \quad \text{for } \tau > 0.$$

Then system [2] has on $[a, T)$ not more than one solution.

Condition 3 is satisfied, for example, for $\varphi(t) = \omega_{-1}(t)$. However, in concrete cases, as $\varphi(t)$ one can usually choose a more rapidly increasing function. Thus, for $\omega(t) = |\ln t|^{-1}$ one may set $\varphi(t) = \exp\{-t^{-\nu}\}$, $0 < \nu \le 1$, and we obtain

Corollary. Suppose that for $t_0 \le s \le t \le t_0 + \varepsilon$, $t_0 \notin P$, $\varepsilon > 0$, and arbitrary x and y $(\|x\| < C,\ \|y\| < C,\ \|x - y\| \le \varepsilon)$:

1. $\|G(t, s, x) - G(t, s, y)\| \le K|\ln\|x - y\||^{-1}$.
2. $t - g(t) \le t_0 + \exp\{-\exp(t - t_0)^{-\nu}\}$, $0 < \nu \le 1$.

Then system (2) has no more than one solution on $[a, T]$.

For $\omega(t) = t^{\alpha}$, $0 < \alpha < 1$, and $t - g(t) \le t_0 + b(t - t_0)^{\gamma(t)}$, by virtue of condition 3, Theorem 1 guarantees uniqueness of the solution only if $\alpha\gamma > 1$. The theorem of A. D. Myshkis ([1]) shows that, if the function G does not depend on t, system (2) has a unique solution also in the case $\gamma = 1/\alpha$ (see also ([2])). Moreover, A. D. Myshkis showed that, whatever $\varepsilon > 0$ is, for $\gamma = 1/\alpha - \varepsilon$ there is always a system (2) having at least two solutions. The assertion given below somewhat refines these results: it turns out that one may set $\gamma = 1/\alpha - \varphi(t - t_0)$, if $\varphi(t)$ tends to zero sufficiently rapidly as $t \to 0$.

Theorem 2. Suppose that for $t_0 \le s \le t \le t_0 + \varepsilon$, $t_0 \notin P$, $\varepsilon > 0$, the following conditions are satisfied:

1. $\|G(t,s,x) - G(t,s,y)\| \le K\|x-y\|^\alpha$ ($\|x\| < C,\ \|y\| < C,\ \|x-y\| \le \varepsilon,\ 0 < \alpha < 1$).
2. $t - g(t) \le t_0 + (t-t_0)^{1/\alpha - \varphi(t-t_0)}$, where $\varphi(t)$ is a nonnegative and nonincreasing function such that, for some positive $\beta < 1/\alpha$ and $\tau \in [0,\varepsilon]$,

$$0 < r - \alpha \sum_{k=1}^{r-1} (r-k)\varphi(\tau^{\beta k - 1}) \to \infty \quad \text{as } r \to \infty.$$

Then system (2) has a unique solution on $[a,T]$.

Corollary. Suppose that for $t_0 \le s \le t \le t_0 + \varepsilon$, $t_0 \notin P$, $\varepsilon > 0$, and arbitrary x and y ($\|x\| < C,\ \|y\| < C,\ \|x-y\| \le \varepsilon$):

1. $\|G(t,s,x) - G(t,s,y)\| \le K\|x-y\|^\alpha$.
2. $t - g(t) \le t_0 + b(t-t_0)^{1/\alpha} |\ln(t-t_0)|^{|\ln(t-t_0)|^\mu}$, where $0 < \alpha < 1$, $b > 0$, $0 \le \mu < 1$.

Then system (2) has a unique solution on $[a,T]$.

We note that the assertions formulated above can be extended to systems with a finite number of delays.

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CITED LITERATURE

[1] A. D. Myshkis, *UMN*, **4**, No. 5, 99 (1949).
[2] A. I. Logunov, *DAN*, **151**, No. 2, 256 (1963).

Note: Figure translations are in progress. See original paper for figures.