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Abstract

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PHYSICS

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NONLINEAR THEORY OF THE MOTION OF FAST CHARGED PARTICLES IN HELICAL TOROIDAL MAGNETIC FIELDS

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In the present paper we consider helical fields wound into a torus, the mean radius of which R is much greater than the radius of the transverse section σ_0; it is assumed that the pitch of the helical field L is greater than the radius σ_0.

Such magnetic systems may be used for strong focusing of charged particles in accelerators and in devices intended for channeling charged particles. In the helical magnetic systems under consideration the field rotates along the trajectory of a fast particle, whereas in existing strongly focusing systems fields with alternating sign of the gradient are used. By averaging methods, adiabatic invariants are constructed which describe nonlinear oscillatory motions analogous to betatron oscillations. The equations describing these oscillations are linearized for small oscillation amplitudes, and formulas for their frequencies are obtained which describe the principal linear resonances of the system. The paper considers the motion of a particle in a helical toroidal field in the presence of a reversing field and without it. The necessary relations connecting the system parameters R, L, σ_0, the magnitude of the helical field H, and the reversing field H_0 are obtained.

Let us introduce coordinates σ, χ, φ as follows: σ and χ are polar coordinates in the transverse section of the torus; φ is the azimuthal angle, measured along an arc of a circle of radius R.

We shall consider fields increasing linearly with distance from the axis of helical symmetry, and shall present results pertaining to the case of nonlinear growth of the field.

The principal harmonic of the potential of a 2-start toroidal helical field is represented by formula (1)

$$\Phi_0 = \left(1 - \frac{\sigma}{2R}\cos\chi\right) I_2\left(\frac{n\sigma}{R}\right)\sin(2\chi - n\varphi). \tag{1}$$

We shall take only the first term in the expansion of the Bessel function of imaginary argument I_2; the smallness of the corrections is determined by the smallness of the ratio $\sigma_0^2 n^2/R^2 = (2\pi\sigma_0/L)^2 = 4\varepsilon_1$, by the rapid convergence of the series for I_2, and also by the fact that we consider motion in the near-axis region, where σ is small.

The components of the helical field together with the reversing field H_0, taking into account perturbations of order $\varepsilon_2 = \sigma_0/R$ caused by the toroidality of the field, will have the form

$$H_\sigma = H\rho\sin\psi\left(1 - \frac{3}{4}\varepsilon_2\rho\cos\chi\right) + H_0\sin\chi + \frac{2}{3}\varepsilon_1 H\rho^3\sin\psi,$$

$$H_\chi = H\rho\left[\cos\psi - \frac{1}{4}\varepsilon_2\rho(2\cos\chi\cos\psi - \sin\psi)\right] + H_0\cos\chi + \frac{1}{3}\varepsilon_1 H\rho^3\cos\psi,$$

$$H_\varphi = -H\rho^2\frac{\pi\sigma_0}{L}\cos\psi, \tag{2}$$

where $\sigma = \sigma_0\rho$; n is the number of revolutions of the helical field on the torus; $\psi = 2\chi - n\varphi$ is the phase of the field at the particle point. The motion of the particle in the plane of the transverse section of the torus is conveniently described with the aid of the variables $v_\perp$ and α, where $v_\perp =$

$$= \sqrt{\gamma\sigma^2 + (\sigma\Delta)^2};$$

α is the angle formed by the vector $v_\perp$ with the direction $\chi = 0$; $v = \sqrt{v_\perp^2 + R^2\dot{\varphi}^2}$ is the total velocity.

A fast particle, i.e., a particle whose velocity is directed primarily along the axis of helical symmetry of the field and, consequently, for which the ratio $v_\perp/v \ll 1$, while passing successively through the periods of the helical field, is in the field of a rapidly rotating force. The rapid rotation of the force means that during one revolution of the field the particle executes certain rapid motions whose amplitude is much smaller than the transverse dimensions of the chamber. The magnitude of this rotating force depends on σ, and therefore it will cause a slow systematic motion of the particle, a drift, which is analogous to betatron oscillations in a strong-focusing accelerator. The natural method for constructing a nonlinear theory of particle motion in such fields will be to average the action of these rapidly alternating forces. The considerations set forth concerning the choice of the parameters of the system determine the smallness of the dimensionless parameter ε

$$\varepsilon = \frac{evH}{cm\xi_0}\left(\frac{L}{2\pi v}\right)^2 = a\left(\frac{L}{2\pi v}\right)^2. \tag{3}$$

In order to put the system of equations of motion into a form convenient for averaging, we introduce the dimensionless time $\tau = t \cdot 2\pi v/L$, which has the meaning of the angle of rotation of the field, the velocity w from the condition $v_\perp = \sqrt{a\varepsilon\sigma_0\rho w}$, the angle $\theta = \psi + \alpha - \chi$, and also the parameter $\delta = L^2/(2\pi)^2 R\sigma_0 - H_0/H$, $d/d\tau = {}^\circ$.

In these variables the equations of motion

$$\frac{d\mathbf{v}}{dt} = \frac{e}{cm}[\mathbf{vH}]$$

will be written in the form

$$\begin{aligned} w\dot{\theta} = & -w + \sin\theta + \varepsilon w^2 \sin(\theta - \psi) + \varepsilon_2 \frac{\rho}{4}[-2\cos\chi\cos\theta - \sin\psi\cos\alpha] \\ & + 2\rho^2 w\varepsilon\varepsilon_1 \cos\psi - \delta\frac{\sin\alpha}{\rho} + \frac{\varepsilon_1}{3}[\sin\theta + \sin\psi\cos(\alpha - \chi)], \end{aligned} \tag{4}$$

$$\begin{aligned} \dot{w} = & -\cos\theta - \varepsilon w^2 \cos(\theta - \psi) + \varepsilon_2 \frac{\rho}{4}[2\cos\chi\cos\theta - \sin\psi\sin\alpha] \\ & + \delta\frac{\cos\alpha}{\rho} + \frac{\varepsilon_1}{3}\rho[-\cos\theta + \sin\psi\sin(\alpha - \chi)], \end{aligned}$$

$$\dot{\psi} = -1 + \varepsilon \cdot 2w\sin(\theta - \psi);$$

$$\dot{\rho} = \varepsilon\rho w\cos(\theta - \psi),$$

$$\dot{\chi} = \varepsilon w\sin(\theta - \psi),$$

$$\ddot{\varphi} = 2\varepsilon^2\varepsilon_1\sqrt{\varepsilon_2\rho w}.$$

In the system it is convenient to retain the equation for χ. In the system (4) we shall call the variables θ, w, ψ fast, and ρ and χ slow, since their derivatives are of order ε. From the last equation it follows that the quantity φ has a very high order of smallness ($\sim \varepsilon^3$), i.e., φ is conserved in the first approximations. This makes it possible to separate the motions, regarding the azimuthal motion as uniform with high accuracy.

The degenerate system ($\varepsilon = 0$)

$$\dot{\psi} = -1, \qquad w\dot{\theta} = -w + \sin\theta, \qquad \dot{w} = -\cos\theta \tag{5}$$

describes the motion of a particle in the field of a rotating force, constant in magnitude, and has the integrals

$$p = w^2 - 2w\sin\theta,$$

$$\Phi = \theta - \psi - \begin{cases} \arcsin\left(\cos\theta/\sqrt{1+p}\right), & \text{for } p > 0, \\ \operatorname{Arcsin}\left(\cos\theta/\sqrt{1+p}\right), & \text{for } p < 0. \end{cases} \tag{6}$$

Following the method of paper ([3]), we shall make a substitution in system (4), choosing as the new variables p, Φ together with the slow variables ρ and χ. As a result of such a substitution we obtain a system of slow motions, for the analysis of which we shall apply the asymptotic theory ([2]).

Averaging of the system for p, Φ, ρ, χ is carried out at constant p and Φ, i.e., along the integral curves of the unperturbed system. Retaining the same notation for the variables, we obtain:

$$\begin{aligned} &\dot\rho = \varepsilon\rho\sqrt{1+p}\cos\Phi, \\ &\dot\chi = \varepsilon\sqrt{1+p}\sin\Phi, \\ &\dot p = -\varepsilon\cdot 2\sqrt{1+p}\,(2+\varepsilon_1\rho^2+p)\cos\Phi + \delta\cdot 2\frac{\sqrt{1+p}}{\rho}\cos(\Phi+\chi), \\ &\dot\Phi = -\varepsilon\,\frac{p-\varepsilon_1\rho^2}{\sqrt{1+p}}\sin\Phi - \delta\,\frac{\sin(\Phi+\chi)}{\rho\sqrt{1+p}}. \end{aligned} \tag{7}$$

Averaging the corrections $\sim \varepsilon_1/3$ in the expansion of $\bar I_2$ gives 0.

If $\varepsilon_1 \sim \varepsilon$, $\delta \sim \varepsilon^2$, or $\delta = 0$, then the corresponding terms may be assigned to the second approximation, and system (7) will have integrals, called adiabatic invariants:

$$(2+p)\rho^2 = \lambda, \qquad \sin^2\Phi\,\rho^2(\lambda-\rho^2) = \gamma. \tag{8}$$

In the Φ, ρ^2 plane, the points $\pm\pi/2, \lambda/2$ are singular points of center type; the integral curves form two families of closed convex smooth figures containing the centers. In the neighborhood of the centers the equations may be linearized, so that ρ^2, Φ, p will perform small harmonic oscillations with frequency

$$\Omega = |eHL|/cm\sigma_0\pi, \tag{9}$$

which does not depend on λ, and hence not on the initial conditions. The variable χ is rotational; in the same approximation $\chi = \varepsilon\tau + \chi_0$.

Thus, in the linear approximation the particle slowly moves along a circle of radius $\sqrt{\lambda/2}$, and on this circumference two periods of small harmonic oscillations fit; in addition, the particle executes fast motions in an ε-neighborhood of the described trajectory.

If the frequency Ω is a multiple of the frequency of revolution of the particle along a circumference of radius R, then a small perturbation of the field may excite the oscillations, as a result of which the oscillations will become nonlinear; however, in the linear approximation we are dealing with a basic linear resonance.

For fields increasing faster than (r^{m-1}, where $m = 3, 4, 5, ...$), the invariants have the form:

$$\rho^2 \left(p + 2\rho^{2(m-2)}\right) = \lambda, \qquad \rho^{2(m-1)} \left(\lambda - \rho^{2(m-1)}\right) \sin^{2(m-1)} \Phi = \gamma. \tag{10}$$

The picture of the motion in such fields will be analogous, with the difference that the frequency Ω will depend on λ, and hence on the initial conditions; moreover, since the energy density of the field near the axis is small and increases sharply toward the periphery, for large m the particles will be focused near circumferences of larger radius.

Let us consider the influence of toroidal distortions. If the twisting field $H_0 = HL^2/\varepsilon(2\pi)^2 R\sigma_0$, $\delta = 0$, system (4) contains terms of order ε_2 that take into account the toroidal distortions of the field. But in the averaged system they are absent. This means that, in the presence of a twisting field H_0 and for $\sigma_0/R \sim \varepsilon$, toroidality does not affect the motion on the average. If the twisting field is absent, one must, mainly for

at the expense of increasing R, to attain the smallness of $\delta_1 = \delta/\varepsilon$, i.e., it is necessary to require

$$\frac{1}{\varepsilon^2} \frac{\sigma_0}{R} \left(\frac{L}{2\pi\sigma_0}\right)^2 < 1. \tag{11}$$

In system (7) one can take into account the influence of small perturbations of order δ_2 if, regarding ρ, χ, p, Φ as fast variables, one introduces into the system the slow variables λ and γ and averages the right-hand sides along the integral curves of the unperturbed system. Owing to the presence of the rotational variable χ, the result obtained will be zero. In the same way one can take into account in system (7) the influence of the component of the field H_φ of order ε_1; on the average, zero will likewise be obtained. Perturbations of the nonlinear pattern of motion are described by introducing into the invariants λ and γ terms of order ε_1 and δ_1.

On the basis of the results obtained, one may propose a new construction for channeling charged particles—a "magnetic hose," based on the use of helical fields. In a helical field, fast particles with a wide range of velocities can be transported; it is only necessary that the parameter ε remain small. With the aid of the adiabatic invariants λ and γ, taking into account the characteristics of the particle source, one can choose the parameters of the system so that particle losses are minimal. The "magnetic hose" can be bent arbitrarily; it is only necessary that the radius of curvature satisfy condition (11). The properties of straight helical fields were investigated by the author in work [4].

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Note: Figure translations are in progress. See original paper for figures.