A. F. LEONT'EV

1965

SovietRxiv

Abstract

Full Text

A. F. LEONT' EV

CONSTRUCTION OF A FUNCTIONAL EQUATION WITH A GIVEN SYSTEM OF PARTICULAR SOLUTIONS

(Presented by Academician I. M. Vinogradov on 22 VI 1964)

In the paper ([1]) a functional equation was constructed with the system of particular solutions $\{f(\lambda_n z)\}$, where $f(z) = \sum_0^\infty a_n z^n$, $a_n \neq 0$, is a fixed entire function, and $\{\lambda_n\}$ is a fixed sequence of complex numbers, $\lim_{n\to\infty} \lambda_n = \infty$. The construction is carried out in the following way and under the following conditions. It is assumed that $f(z)$ is an entire function of order ρ and type σ, satisfying the condition

$$\lim_{n\to\infty} n^{1/\rho} \sqrt[n]{|a_n|} = (\sigma e \rho)^{1/\rho} > 0,$$

and that $\{\lambda_n\}$ is such that

$$\overline{\lim}_{n\to\infty} \frac{n}{|\lambda_n|^\rho} = \tau < \infty.$$

Let m be an integer $> \rho$ and

$$L(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z^m}{\lambda_n^m}\right).$$

The function $L(z)$ is an entire function growing no faster than an entire function of order ρ of finite type. Put

$$L(\eta) f(z\eta) = \sum_{n=0}^{\infty} B_n(z) \eta^n, \qquad A_n(z) = \frac{B_n(z)}{a_n}, \qquad \psi(z,t) = \sum_{n=0}^{\infty} \frac{A_n(z)}{t^{n+1}}.$$

The function $\psi(z,t)$ is regular for sufficiently large $|t|$. For example, if $\underline{\sigma} = \sigma$ and $L(z)$ is of order ρ and type σ_1, then $\psi(z,t)$ is regular in the domain

$$|z| < r, \qquad |t| > \left(\frac{\sigma_1}{\sigma} + r^\rho\right)^{1/\rho}.$$

Next put

$$M(F) = \frac{1}{2\pi i}\int_C \psi(z,t)F(t)\,dt,$$

where C is the circle $|t| = R$, chosen so that, for $|z| < r$, the function $\psi(z,t)$, as a function of the variable t, is regular on C. The function $F(t)$ on the contour C and inside the contour C is assumed to be analytic. The equality

$$M[f(\lambda z)] = L(\lambda)f(\lambda z),$$

holds, from which it follows that the functions $f(\lambda_n z)$ $(n = 1, 2, ...)$ are particular solutions of the equation

$$\frac{1}{2\pi i}\int_C \psi(z,t)F(t)\,dt = 0. \tag{1}$$

This last equation is precisely the functional equation mentioned above. In the paper ([1]) and in a number of other works it was used to study properties of functions $F(z)$ which, in a domain D where the system $\{f(\lambda_n z)\}$ is not complete, are approximated with arbitrary accuracy by finite linear combinations of functions from the system $\{f(\lambda_n z)\}$.

In the article a functional equation is constructed with a more general system of particular solutions. Let $A(z,h)$ be some function of two variables z and h, analytic in the variable h for any finite h, when z belongs to some set D. We have

$$A(z,h) = \sum_{m=0}^{\infty} P_m(z)h^m, \quad z \in D,\ |h| < \infty. \tag{2}$$

We assume that $P_n(z) \not\equiv 0$ $(n = 0, 1, 2, ...)$ on the set D. We note that the functions $P_n(z)$ are not necessarily analytic. We shall construct an equation with the system of particular solutions $\{A(z, \lambda_n)\}$. For definiteness, suppose that for each fixed $z \in D$ the function $A(z,h)$ is an entire function of order ρ and finite type σ, $\sigma = \sigma_z$. Let $\{a_k\}$ be some sequence of nonzero complex numbers such that there exists the finite limit

$$\lim_{n\to\infty} n^{1/\rho}\sqrt[n]{|a_n|} > 0. \tag{3}$$

Take any entire function $L(\lambda)$ of order not exceeding ρ and of finite type. Put

$$L(\eta)A(z,\eta) = \sum_{n=0}^{\infty} B_n(z)\eta^n, \quad A_n(z) = \frac{B_n(z)}{a_n}, \quad \psi(z,t) = \sum_{n=0}^{\infty}\frac{A_n(z)}{t^{n+1}}.$$

By virtue of the assumptions made, the function $\psi(z,t)$, when z is fixed from the set D, is regular for sufficiently large $|t|: \ |t| > \mu_z$. Denote by K the following class of functions: a function $F(z) \in K$ if on some part G of the set D the function $F(z)$ is represented in the form

$$F(z) = \sum_{n=0}^{\infty} b_n P_n(z). \tag{4}$$

To a function $F(z)$ represented by the series (4), assign the function

$$F_1(z) = \sum_{n=0}^{\infty} b_n a_n z^n. \tag{5}$$

Suppose that the series (5) converges in the disk $|z| < R_0, \ R_0 > 0$. We note that, according to the indicated rule, to the function $F(z) = A(z,h)$ there corresponds the function $F_1(z) = f(zh)$, where

$$f(z) = \sum_{0}^{\infty} a_n z^n. \tag{6}$$

Introduce the operator

$$M(F) = \frac{1}{2\pi i} \int_{|t|=R} \psi(z,t) F_1(t)\, dt, \quad F \in K. \tag{7}$$

We assume that R can be chosen so that the condition

$$\mu_z < R < R_0$$

is satisfied for some z from the set G. Let us see into what the operator (7) transforms the function $F(z) = A(z,h)$. To the function $A(z,h)$, as was already noted, there corresponds the function $F_1(z) = f(hz)$. But

$$\frac{1}{2\pi i} \int_{|t|=R} \psi(z,t) f(ht)\, dt = \sum_{n=0}^{\infty} A_n(z) a_n h^n = L(h) A(z,h).$$

Therefore

$$M[A(z,h)] = L(h) A(z,h). \tag{8}$$

Let $\{\lambda_n\}$ be some sequence of complex numbers with the property

$$\lim_{n\to\infty} \frac{n}{|\lambda_n|^\rho} = \tau < \infty.$$

Choose the function $L(\lambda)$ so that it vanishes on the sequence $\{\lambda_n\}$. It then follows from relation (8) that $A(z, \lambda_n)$ $(n = 1, 2, ...)$ are particular solutions of the equation

$$M(F) = 0. \tag{9}$$

The last equation is precisely the required functional equation.

Equation (9), where the operator is defined by formula (7), has the essential drawback that it depends on the function $F(z)$ through the function $F_1(z)$, i.e., it depends on $F(z)$ in an implicit way. We indicate a case in which this drawback can be eliminated. Suppose that the set D is a simply connected domain, the functions $P_n(z)$ are regular in D, and the system $\{P_n(z)\}$ possesses in D a biorthogonal system $\{\mu_n(t)\}$:

$$\frac{1}{2\pi i} \int_C P_n(t)\mu_m(t)\, dt = \begin{cases} 0, & n \neq m, \\ 1, & n = m. \end{cases}$$

Here C is some closed contour belonging to the domain D, and the functions $\mu_n(z)$ are regular on the contour C and outside it. If the function $F(z)$, defined by formula (4), is regular on the contour C (more precisely, if the series (4) converges uniformly on the contour C), then, obviously,

$$b_n = \frac{1}{2\pi i} \int_D F(\xi)\mu_n(\xi)\, d\xi.$$

Hence we conclude that the function (5) has the form

$$F_1(t) = \frac{1}{2\pi i} \int_C F(\xi) q(t, \xi)\, d\xi, \qquad q(t, \xi) = \sum_{n=0}^{\infty} a_n t^n \mu_n(\xi).$$

In the case $P_k(z) = a_k z^k$ we have $\mu_k(z) = 1/a_k z^{k+1}$, and therefore

$$q(t, \xi) = \sum_{n=0}^{\infty} \frac{t^n}{\xi^{n+1}} = \frac{1}{\xi - t},$$

the Cauchy kernel. In this case $F_1(t) = F(t)$. Suppose that the subsequent operations (change of order of integration, integration of a sum) are valid. We obtain

$$\frac{1}{2\pi i}\int_{|t|=R}\psi(z,t)F_1(t)\,dt = \frac{1}{2\pi i}\int_{|t|=R}\psi(z,t)\frac{1}{2\pi i}\int_C F(\xi)q(t,\xi)\,d\xi =$$

$$= \frac{1}{2\pi i}\int_C F(\xi)\left\{\frac{1}{2\pi i}\int_{|t|=R}\psi(z,t)q(t,\xi)\,dt\right\}d\xi.$$

Denote the inner integral by $\gamma(z,\xi)$. We have

$$\gamma(z,\xi) = \sum_{n=0}^{\infty} A_n(z)\frac{1}{2\pi i}\int_{|t|=R}\frac{q(t,\xi)\,dt}{t^{n+1}} = \sum_{n=0}^{\infty} a_n A_n(z)\mu_n(\xi)$$

or

$$\gamma(z,\xi) = \sum_{n=0}^{\infty} B_n(z)\mu_n(\xi). \tag{10}$$

Thus,

$$M(F) = \frac{1}{2\pi i}\int_C \gamma(z,\xi)F(\xi)\,d\xi, \tag{11}$$

where the kernel $\gamma(z,\xi)$ is determined by formula (10). After we have formally obtained the representation (11), let us make some assumptions. Suppose that the series (10) converges uniformly with respect to the variable ξ when $\xi \in C$, while z is fixed. Further, let the function $F(\xi)$ be analytic on the contour C and inside C. Under these conditions expression (11) makes sense. Let us verify that relation (8) holds. We have

$$M[A(z,h)] = \frac{1}{2\pi i}\int_C \gamma(z,\xi)A(\xi,h)\,d\xi = \sum_{n=0}^{\infty} B_n(z)\frac{1}{2\pi i}\int_C A(\xi,h)\mu_n(\xi)\,d\xi.$$

Suppose that the series (2) converges uniformly when h is fixed and z varies on the contour C. Then we obtain

$$M[A(z,h)] = \sum_{n=0}^{\infty} B_n(z)\sum_{m=0}^{\infty} h^m \frac{1}{2\pi i}\int_C P_m(\xi)\mu_n(\xi)\,d\xi =$$

$$= \sum_{n=0}^{\infty} B_n(z)h^n = L(h)A(z,h).$$

Consequently, the operator (11) is precisely the operator we were seeking.

The operator (7), or the operator (11), can also be represented differently. To this end, let us consider the special case when $L(\eta) = \eta^m$. In this case

$$B_n(z) = \begin{cases} 0, & n < m, \\ P_{n-m}(z), & n \geq m, \end{cases}$$

and, consequently,

$$\psi(z,t) = \sum_{n=m}^{\infty} \frac{P_{n-m}(z)}{a_n t^{n+1}},$$

whence

$$\frac{1}{2\pi i} \int_{|t|=R} \psi(z,t) F_1(t)\, dt = \sum_{n=m}^{\infty} b_n P_{n-m}(z).$$

Put

$$D^m F = \sum_{n=m}^{\infty} b_n P_{n-m}(z).$$

In the general case

$$L(\eta) = \sum_{m=0}^{\infty} C_m \eta^m,$$

and therefore

$$M(F) = \sum_{m=0}^{\infty} C_m D^m F. \tag{12}$$

In the case of equation (11) we have

$$D^m F = \frac{1}{2\pi i} \int_C \gamma_m(z,\xi) F(\xi)\, d\xi,$$

where

$$\gamma_m(z,\xi) = \sum_{n=m}^{\infty} P_{n-m}(z) \mu_n(\xi).$$

The operators D^{mF} have the following properties:

$$D^{n+m}F = D^n(D^{mF}), \qquad D^m A(z,h) = h^m A(z,h).$$

If $A(z,h) = e^{zh}$, then it is easy to see that D^{mF} coincides with the ordinary derivative $F^{(m)}(z)$. In the case $A(z,h) = f(zh)$, the operators D^{mF} were introduced in paper [2].

The operators (7), (11), (12) may be used in the study of functions $F(z)$ which, in some domain G, are approximated with arbitrary accuracy by means of the system $\{A(z,\lambda_n)\}$. The system $\{A(z,\lambda_n)\}$ is assumed to be incomplete in the domain G.

Steklov Mathematical Institute
Academy of Sciences of the USSR

Received
15 VI 1964

CITED LITERATURE

[1] A. F. Leont' ev, Tr. Matem. inst. im. V. A. Steklova AN SSSR, **39** (1951).

[2] A. O. Gelfond, A. F. Leont' ev, Matem. sborn., **29** (71), 3, 477 (1951).

Note: Figure translations are in progress. See original paper for figures.