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Abstract

Full Text

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ON THE POINTWISE AND INTEGRAL CARDINALITY OF SYSTEMS OF SUBSETS OF A TOPOLOGICAL SPACE

(Presented by Academician P. S. Aleksandrov on 24 X 1964)

Corson and Michael proved [1] that a point-countable base of a countably compact space is countable. Here a stronger assertion is proved (Theorem 1). Theorem 2, in particular, answers the question of how the pointwise and integral cardinalities of a base of a finally compact space are related.

Definition. A system of open sets is called a t -pseudobase of a space X if, for every point $x \in X$, it contains a subsystem whose elements have intersection exactly the point x .

A topological space possessing a t -pseudobase is automatically a T_1 -space. A base of a T_1 -space is, evidently, a t -pseudobase.

Theorem 1. *Let X be a countably compact space, and let B be its t -pseudobase, point-countable on an everywhere dense subset M of X . Then the t -pseudobase B is countable.*

Proof. We shall construct by induction countable sets C_i for all natural i . Take an arbitrary point $a \in M$ and its countable star from the t -pseudobase B ; denote the star by γ . For every finite collection of elements $U_1, U_2, \dots, U_m \in \gamma$, choose one point

$$x(U_1, \dots, U_m) \in M \cap \left(X \setminus \bigcup_{j=1}^m U_j \right),$$

if such a point exists for the given collection. Denote by C_1 the union of such points (over all finite collections of the star γ). Suppose C_i has been constructed. We shall construct C_{i+1} . The system B_i is the union of all those elements of the t -pseudobase B which intersect at least one of the C_k , $k = 1, \dots, i$. If $U_1, \dots, U_m \in B_i$ (where m is a natural number), then take a point

$$x(U_1, \dots, U_m) \in M \cap \left(X \setminus \bigcup_{j=1}^m U_j \right),$$

if such a point is found. The union of all such points, taken one for each finite collection of elements of B_i , is the set C_{i+1} .

Each of the sets C_i is countable. Put $C = \bigcup_{i=1}^{\infty} C_i$; evidently, the set C is countable. We show that $[C] = X$. Suppose the contrary, $X \setminus [C] \neq \Lambda$. Then there is a point $a \in M \cap (X \setminus [C])$. For each point $x \in [C]$ there is an element $U_\alpha \in B$, containing the point x and such that $U_\alpha \not\ni a$, since B is a t -pseudobase in X . The union of such U_α over all points $x \in [C]$ forms a cover $\omega = \{U_\alpha\}$ of the set $[C]$. The cover ω is countable, since C is a countable everywhere dense subset of $[C]$ and since each point of the set $C \subseteq M$, by the hypothesis of the theorem, belongs to no more than countably many elements of B .

Since $[C]$, as a closed subspace of a countably compact space, is countably compact, from the countable cover ω one can choose

finite cover $\{U_1, \dots, U_m\}$; $[C] \subseteq \bigcup_{j=1}^m U_j$. It is not hard to see that there is a natural number i such that $U_1, U_2, \dots, U_m \in B_i$. But then C_{i+1} must contain the point

$$x(U_1, U_2, \dots, U_m) \in M \cap \left(X \setminus \bigcup_{j=1}^m U_j \right),$$

if such a point exists. But such a point does exist: for example, the point a . Thus, a point of C_{i+1} has been found which belongs to $X \setminus \bigcup_{j=1}^m U_j$, which contradicts the fact that $C_{i+1} \subseteq C \subseteq \bigcup_{j=1}^m U_j$. Hence, $[C] = X$, i.e. C is an everywhere dense countable set of the space X .

Every point of the set $C \subseteq M$ belongs to at most countably many elements of B , and every element of B contains at least one point of the set C ; hence it follows that the t -pseudobase B is countable, as was required to prove.

Theorem 1 has the following generalization.

Theorem 1'. *Let X be a space such that, from every open cover of cardinality $\tau \leq \aleph_0$, one can extract a finite cover, and let B be its t -pseudobase of pointwise cardinality $\leq \tau$ on an everywhere dense subset M of X . Then the t -pseudobase B has cardinality $\leq \tau$.*

Corollary 1. *Let X be a bicomact space, and let B be its t -pseudobase whose pointwise cardinality is $\leq \tau$ on an everywhere dense subset M of X . Then the cardinality of $B \leq \tau$.*

This corollary includes the theorem of A. Mishchenko ⁽²⁾ (together with the strengthening given to it in ⁽³⁾). From Theorem 1 there follows the corresponding theorem of Corson and Michael from ⁽¹⁾.

Corollary 2. *Let X be a locally bicomact Hausdorff space, and let B be its base, pointwise countable on an everywhere dense subset M of X . Then X is metrizable.*

Corollary 3. *Let X be a locally bicomact Hausdorff space, and let B be its base whose pointwise cardinality is $\leq \tau$ on an everywhere dense subset M of X .*

Then X decomposes into isolated subspaces of weight $\leq \tau$. In particular, if X is connected, then its weight is $\leq \tau$.

We derive Corollary 3. From B we select a subsystem B' of such elements whose closures are bicomact. The subsystem B' , obviously, is a cover of X , the elements of which have weight $\leq \tau$, in view of Corollary 1. We shall call points $x_0, x_1 \in X$ equivalent if there exists a finite set of elements $B_0, \dots, B_n \in B'$ such that $x_0 \in B_0, x_1 \in B_n, B_i \cap B_{i+1} \neq \Lambda, i = 1, 2, \dots, n - 1$.

It is easy to see that X decomposes into open subspaces of equivalent points. Denote one of them by A . Since each element of the cover B' intersects no more than τ other elements, the locally bicomact subspace A is the union of no more than τ subsets whose weights are $\leq \tau$. By the addition theorem it follows that the weight of $A \leq \tau$. The assertion is proved.

Examples are known ^(1,2) of finally compact spaces with a pointwise countable base, but without a countable base. However, the following theorem is true, limiting the cardinality of pointwise countable bases.

Theorem 2. *Let X be a finally compact space, and let B be its t -pseudobase, pointwise countable on an everywhere dense subset M of X . Then the cardinality of the t -pseudobase $B \leq 2^{\aleph_0}$.*

Sketch of the proof. The proof of Theorem 2 has much in common with the proof of Theorem 1. We shall construct, by transfinite induction, sets C_α for all ordinal numbers α up to ω_1 exclusively (up to ω_1 because no countable sequence is cofinal with it—

ness). I take an arbitrary point $a \in M$ and its countable star γ from the t -pseudobase B . For every countable set of elements $\{U_i\} \in \gamma$ choose one point

$$x\{U_i\} \in M \cap \left(X \setminus \bigcup_{i=1}^{\infty} U_i \right),$$

if such a point exists. Denote by C_1 the union of such points for all countable (or finite) sets of the star γ . Suppose that C_α have already been constructed for all ordinal numbers α up to the ordinal number μ . We shall construct C_μ . The system B_μ is the union of all those elements of the t -pseudobase B which intersect at least one of the C_α , where $\alpha < \mu$. Let a countable subsystem $\{U_i\} \in B_\mu$; then take a point

$$x\{U_i\} \in M \cap \left(X \setminus \bigcup_{j=1}^{\infty} U_j \right),$$

if such a point exists. The union of such points, taken one for each countable (or finite) set of elements of B_μ , is the set C_μ . Each of the sets C_α has cardinality $\leq 2^{\aleph_0}$. Put $C = \bigcup_\alpha C_\alpha$; obviously, the cardinality of $C \leq 2^{\aleph_0}$.

It remains to prove that $[C] = X$, after which one may conclude that the cardinality of the t -pseudobase $B \leq 2^{\aleph_0}$.

Theorem 2 has the following generalization.

Theorem 2'. *Let X be a space such that from every open cover of cardinality 2^τ ($\tau \geq \aleph_0$) one can select a cover of cardinality $\leq \tau$, and let B be its t -pseudobase of point cardinality $\leq \tau$ on an everywhere dense subset M of X . Then the t -pseudobase B has cardinality $\leq 2^\tau$.*

For $\tau = \aleph_0$, Theorem 2' gives a stronger assertion than Theorem 2.

We now give a theorem on the cardinality of t -pseudobases in metric spaces.

Theorem 3. *A metric space of weight $\leq 2^\tau$ has a t -pseudobase of cardinality $\leq \tau$. In particular, a metric space whose weight is 2^{\aleph_0} has a countable t -pseudobase.*

We shall preface the proof of the theorem with several lemmas.

Lemma 1. *Suppose there is a condensation $f : X \rightarrow Y$, where Y has a t -pseudobase of cardinality τ . Then X also has a t -pseudobase of cardinality τ .*

Indeed, if $\omega = \{\omega_\alpha\}$ is a t -pseudobase of cardinality τ in Y , then $f^{-1}\omega = \{f^{-1}\omega_\alpha\}$ is a t -pseudobase in X , in view of the fact that f is a condensation.

Lemma 2. *The bicomactum I^τ (the product of τ copies of a line segment) has a t -pseudobase of cardinality τ .*

On each segment factor I_α there is a countable base $\omega_\alpha = \{U_{\alpha i}\}$; a typical element of the t -pseudobase is the product of $U_{\alpha i}$ with all the remaining segment factors (except I_α).

Lemma 3. *A space of isolated points of cardinality $\leq 2^\tau$ has a t -pseudobase of cardinality $\leq \tau$.*

A space of isolated points of cardinality 2^τ condenses onto I^τ . Since I^τ , by Lemma 2, has a t -pseudobase of cardinality τ , the space of isolated points, by Lemma 1, also has a t -pseudobase of cardinality τ .

We prove Theorem 3. By Bing's criterion, the metric space M has a σ -discrete base $\gamma = \{\gamma_i\}$, where each γ_i is a discrete system of elements; $\gamma = \{\gamma_{i\alpha}\}$. The system γ_i is isomorphic to a space of isolated points of cardinality $\leq 2^\tau$. Consequently, by Lemma 3, there is such a collection of subsystems in γ_i of cardinality $\leq \tau$ that every element of

γ_i can be obtained as the intersection of certain elements of this family; denote this family by Γ_i . It is not hard to verify that $\Gamma = \{\Gamma_i\}$, $i = 1, 2, \dots$, is a t -pseudobase in M . The theorem is proved.

Remark. The cardinality of a space having a countable t -pseudobase is $\leq 2^{\aleph_0}$. Indeed, every point, by the definition of a t -pseudobase, can be obtained as the

intersection of some subsystem of elements of the t -pseudobase, and the set of subsystems of a countable system is $\leq 2^{\aleph_0}$. Hence it follows that Theorem 3 is final.

In conclusion we pose a question. If a bicom pactum has a countable t -pseudobase, does it also have a countable base? In general, does the minimal cardinality of a t -pseudobase of a bicom pactum coincide with its weight?*

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References

1. H. H. Corson, E. Michael, Illinois J. Math., 8, No. 2, 351 (1964).
2. A. Mishchenko, DAN, 144, No. 5, 985 (1962).
3. V. Proizvolov, DAN, 154, No. 1, 55 (1964).

* **Note added in proof.** After the article had been submitted for typesetting, an affirmative answer to this question was obtained.

Note: Figure translations are in progress. See original paper for figures.

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