ON THE RADIATION OF A FINITE VOLUME OF NONEQUILIBRIUM PLASMA

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ON THE RADIATION OF A FINITE VOLUME OF NONEQUILIBRIUM PLASMA

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1. For a number of problems in plasma physics, the question of radiative losses of a finite volume of plasma in spectral lines is essential, with simultaneous allowance for the nonequilibrium population of levels and for reabsorption of radiation ([1–5]). Two somewhat different approaches have been used: a) one based on the direct calculation of the flux of outgoing quanta $Q(\omega)d\omega$ ([1]); b) one based on finding the intensity $I_\omega(\mathbf{r})$ by the usual theory of radiative transfer ([2–4]).

The aim of the present work is to show that, for the problem of radiative losses of a bounded volume, the somewhat less rigorous approach used in ([1]), if properly developed, makes it possible to give a simpler, mathematically compact, and physically transparent treatment. The point is that in this problem it is usually sufficient to find the *integral* flux of outgoing quanta N, and for this quantity simple formal solutions convenient for analysis can be obtained.

2. Let us consider the transfer of isotropic radiation in a two-level model under the usual assumptions of spatial uniformity of the temperature T, of the densities of electrons n_e and atoms in the lower level n_0, and hence of the absorption coefficient $\varkappa(\omega)$. The uniformity of n_0 is ensured when $kT \ll \hbar\omega_0$ (ω_0 is the frequency at the center of the line). Under the same condition induced emission may be neglected. We have (cf. ([1,6,7]))

$$Q(\omega)d\omega = P(\omega)d\omega \int_V d\mathbf{r}' \, \frac{n_1(\mathbf{r}')}{\tau} \left\{ 1 - \int_V d\mathbf{r} \, e^{-\varkappa(\omega)|\mathbf{r}-\mathbf{r}'|} \frac{\varkappa(\omega)}{4\pi|\mathbf{r}-\mathbf{r}'|^2} \right\}, \quad (1)$$

where $n_1(\mathbf{r})$, the density of excited atoms, satisfies the equation

$$\frac{\partial n_1(\mathbf{r})}{\partial t} = n_0 n_e \langle v\sigma_{01} \rangle - n_1(\mathbf{r}) n_e \langle v\sigma_{10} \rangle - \frac{n_1(\mathbf{r})}{\tau} + \int_V \frac{n_1(\mathbf{r}')}{\tau} \, G(\mathbf{r}', \mathbf{r}) d\mathbf{r}'. \quad (2)$$

Here V is the volume of the radiating system; σ_{01}, σ_{10} are the cross sections for excitation and quenching by electron impact; τ is the mean lifetime of an excited atom with respect to spontaneous emission; $P(\omega)$ is the emission-line profile normalized to unity; $G(\mathbf{r}', \mathbf{r})$ is the probability that a quantum emitted at one of the points $\mathbf{r}, \mathbf{r}'$ will be absorbed in a unit volume near the other:

$$G(\mathbf{r}', \mathbf{r}) = -\frac{1}{4\pi|\mathbf{r}' - \mathbf{r}|^2} \frac{dT(|\mathbf{r}' - \mathbf{r}|)}{d|\mathbf{r}' - \mathbf{r}|}, \tag{3}$$

where $T(\rho)$ is the "weighted" probability that a quantum traverses the distance ρ without a single absorption event:

$$T(\rho) = \int_0^\infty P(\omega) e^{-\varkappa(\omega)\rho} d\omega. \tag{4}$$

Introducing the dimensionless function $y(\mathbf{r}) \equiv n_1(\mathbf{r})/n_1^B$, where n_1^B is the Boltzmann density of excited atoms, we reduce (2), for the stationary case,

reducing to the form

$$(1 + \beta) y(\mathbf{r}) = \int_V G(\mathbf{r}, \mathbf{r}') y(\mathbf{r}') \, d\mathbf{r}' + \beta, \tag{5}$$

where $\beta \equiv n_e \langle v\sigma_{10} \rangle \tau$ is the ratio of the probabilities of a quenching collision and spontaneous emission.

3. For the integral flux of outgoing quanta we find

$$N \equiv \int_0^\infty Q(\omega) \, d\omega = (n_1^B/\tau) \int_V y(\mathbf{r}) \, \overline{T}(\mathbf{r}) \, d\mathbf{r}, \tag{6}$$

where $\overline{T}(\mathbf{r})$ is the angularly averaged probability that a quantum emitted from point $\mathbf{r}$ escapes beyond the system without a single absorption event:

$$\overline{T}(\mathbf{r}) = \frac{1}{4\pi} \int_{(4\pi)} T\left[\rho_{\Omega'}^{(S)}\right] \, d\Omega' \tag{7}$$

($\rho_{\Omega'}^{(S)}$ is the distance from point $\mathbf{r}$ to the surface along the ray $\vec{\Omega}'$).

In the stationary case we obtain one more expression for N:

$$N = (\beta n_1^B/\tau) \int_V [1 - y(\mathbf{r})] \, d\mathbf{r} = n_0 n_e \langle v\sigma_{01} \rangle \int_V [1 - y(\mathbf{r})] \, d\mathbf{r}. \tag{8}$$

Expressions (6) or (8), together with equation (5), give the complete solution of the problem. It turns out that even a cursory examination of them makes it possible to go quite far without introducing specific assumptions about the geometry of the system and about the shape of the emission and absorption lines –the latter because the entire dependence on $P(\omega)$ and $\chi(\omega)$ is concentrated in the function $T(\rho)$. Moreover, since in (6) and (8) $y(\mathbf{r})$ enters only under the integral sign, a number of results can be obtained practically without resorting to solving the integral equation (5) itself.

4. Let us investigate the dependence of N on the dimensions of the system. Expressions (6) and (8) admit (in regions that are not mutually exclusive) a simplification of two types:

A. **Volume radiation.** As is seen from (8), in the case of spatial homogeneity $[1-y(\mathbf{r})]$, $N \propto V$, i.e., the radiation is volumetric. A necessary and sufficient condition for the applicability of this approximation is that at least one of the two conditions $T(a) \approx 1$ and $\beta \ll T(a)$ be satisfied, where a is the characteristic size of the system. These conditions can be obtained, even without solving equation (5), directly from the exact relation

$$\int_V [\beta + \overline{T}(\mathbf{r})]\, y(\mathbf{r})\, d\mathbf{r} = \beta V, \tag{9}$$

which follows from (5) and (3). Thus, for an optically thin system ($T(a) \approx 1$), from (9) and (6) we find: $N \approx (\beta/(\beta+1))\, n_1^B V/\tau$, and for a system of arbitrary optical thickness with $\beta \ll T(a)$, $N \approx n_0 n_e \langle v\sigma_{01}\rangle V$. Consequently, even an optically thick plasma ($T(a) \ll 1$), for sufficiently small β (for example, for small n_e), radiates like an optically thin emitter. This result, previously established (by another, considerably more complicated method) in ([3,4]), is obtained here in the most direct way, which proves to be naturally connected with the transparency of its interpretation. Indeed, the inequality $\beta \ll T(a)$, i.e. $n_e\langle v\sigma_{10}\rangle \tau/T(a) \ll 1$, expresses the smallness of the number of quenching collisions over the effective lifetime of an excitation in the system, caused by the repeated re-emission of quanta during their diffusion to the surface of the system. Owing to this smallness, almost every quantum born inside the volume eventually leaves the system, which corresponds to the volume character of the radiation. Thus, the true cause of the (more usual) surface char-

of the nature of the radiation of optically thick systems is not strong absorption of quanta within the volume in itself, but rather sufficiently intense **emission** in the predominant part of the volume.

B. Radiation for a homogeneous source. As is seen from (5), the integral term is relatively small and, consequently, there exists a spatially homogeneous solution $y \approx \beta/(\beta+1)$ in two cases: $\beta \gg 1$ and $T(a) \approx 1$; the fulfillment of at least one of these two conditions is necessary and sufficient. Then (6) can be transformed to the form

Fig. 1

Figure 1: Fig. 1

$$N \approx \frac{\beta}{\beta+1}\,\frac{n_1^{\mathrm{B}}}{\tau}\,\frac{1}{4\pi}\oint dS\int_{(2\pi)} d\Omega\cos\theta\int_0^{\infty}\frac{P(\omega)}{\varkappa(\omega)}\left[1-e^{-\varkappa(\omega)l_\Omega}\right]d\omega. \tag{10}$$

Here dS is an element of the surface of the system; $d\Omega$ is an element of the "outgoing" solid angle with vertex at dS; θ is the angle between the normal to dS and the ray $\vec{\Omega}$; l_Ω is the size of the system along the ray $\vec{\Omega}$.

Fig. 1

In the limit $\varkappa_0 a \ll 1$ ($\varkappa_0$ is the absorption coefficient at the center of the line), (10) reduces to the result for volume radiation (see subsection A). In the opposite limit $\varkappa_0 a \gg 1$, under the usual assumption that the source function is independent of frequency ([7],[8]),

$$\frac{P(\omega)}{\varkappa(\omega)} \approx \frac{g_0}{g_1}\,\frac{\tau}{n_0}\,\frac{\omega_0^2}{\pi^2 c^2}, \tag{11}$$

(10) is brought to the well-known form (here, in addition, $\beta \gg 1$)

$$N \approx \frac{c}{4\pi}\,f_{\mathrm{pl}}(\omega_0)\oint dS\int_{(2\pi)} d\Omega\cos\theta\,\Delta\omega_{\mathrm{eqv}}(l_\Omega), \tag{12}$$

where $f_{\mathrm{pl}}(\omega_0)$ is the Planck (more precisely, Vinograd) density of the number of photons per unit frequency interval near ω_0, and $\Delta\omega_{\mathrm{eqv}}(l)$ is the "equivalent width" of the line. Since $\Delta\omega_{\mathrm{eqv}}$ grows with increasing a ($\sim \sqrt{a}$ for a dispersion profile and $\sim \sqrt{\ln \varkappa_0 a}$ for a Doppler profile), with increasing size of the system N increases faster than simply in proportion to its surface area. As $a \to \infty$, when the inequality $\Delta\omega_{\mathrm{eqv}}(a) \ll kT/\hbar$ is violated, the analysis carried out is formally invalid, but it is not difficult to generalize it using Kirchhoff' s law, which is certainly applicable in this region. Then for the total radiated energy $\mathcal{E}$ we obtain the Stefan–Boltzmann law: $\mathcal{E} = \sigma T^4 S^*$.

For a given $T(a) \ll 1$, formula (12) proves to be approximately valid also when a condition weaker than $\beta \gg 1$ is fulfilled, namely, when $\beta \gg T(a)$; in this case even a significant nonuniformity (a decrease near the surface) of the function $y(\mathbf{r})$, which occurs for $\beta \ll 1$, turns out to be **integrally** small in (6).

5. The results of § 4 are conveniently summarized by means of the diagram (Fig. 1). An interpretation of the problem that does not depend on the specific line shape is based on the parameters β and $T(a)$, but for greater clarity we use the parameters β and $\varkappa_0 a$; for $\varkappa_0 a \ll 1$, $T(a) \approx 1$, while for $\varkappa_0 a \gg 1$, $T(a) \approx (\pi\varkappa_0 a)^{-1/2}$ for a dispersion profile and

$T(a) \approx (\varkappa_0 a\sqrt{\pi \ln \varkappa_0 a})^{-1}$ for a Doppler profile. The diagram is purely schematic: the scale is logarithmic in both directions; the boundary lines of the regions correspond to relative deviations of order unity from the corresponding limiting expressions.

* In the region $\Delta\omega_{\text{eqv}} \ll kT/\hbar$ we have $\mathcal{E} \approx \hbar\omega_0 \cdot N$.

Curve I is described by the equation $\beta = T(a)$. Below it lies the region of strongly nonequilibrium population. Here damping plays no role, so the radiation has the character of an "instantaneous" volume glow after electronic excitation. Above curve I lies the region of equilibrium (Boltzmann) population. Here the role of damping is large, and therefore—depending on the optical thickness $\chi_0 a$—both volume and surface radiation are possible (the corresponding regions are separated by curve II).

Curve II is described by the equation $\beta \approx \frac{1}{2}[(\chi_0 a)^\nu - 1]^{-1}$, where $\nu = 1/2$ for a dispersion contour and $\nu \approx 1$ for a Doppler contour. To the left of it lies the region of volume radiation (in turn divided by curve I into regions of Boltzmann and "instantaneous" emitters), and to the right—the region of surface (blackbody) radiation.

For an approximate judgment as to the volume or surface character of the radiation of a system, the characteristic size a should be compared not simply with the photon mean free path before absorption, $l_a \sim 1/\chi_0$, but with a more complicated length—the mean free path before damping, l_q. An estimate for l_q can be obtained by inverting the equation of curve II:

$$l_q \approx \left(1 + \frac{1}{2\beta}\right)^{1/\nu} \frac{1}{\chi_0}, \tag{13}$$

so that, for example, when $\beta \ll 1$, $l_q \gg l_a$.

Thus, in all, we have three main parameter regions shown in the diagram; in each of them the corresponding limiting formula from Sec. 4 is approximately valid. Near the curves I and II themselves, on segments A and B, the situation is also not complicated, since here the more general formulas of Sec. 4 are applicable: $N \approx (\beta/\beta+1)n_1^{\mathrm{B}} V/\tau$ and (10), respectively. On segment C, however, one must be content with an estimate: here the limiting formulas $N \approx n_0 n_e \langle v\sigma_{01}\rangle V$ and (12) give results of the same order of magnitude, while the true radiation is approximately half as large.

To establish more exact boundaries of the regions, it is necessary to compute corrections to the corresponding limiting formulas. The numerical values of the corrections depend on the line shape and the geometry of the system, and to find them one must solve equation (5), for example by the method proposed in [9].

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Note: Figure translations are in progress. See original paper for figures.