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Abstract

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MATHEMATICS

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FREQUENCY CONDITIONS FOR ABSOLUTE STABILITY AND DISSIPATIVITY OF CONTROLLED SYSTEMS WITH ONE DIFFERENTIABLE NONLINEARITY

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1°. Consider the system*

$$dx/dt = Px + q\varphi(\sigma), \qquad \sigma = r^* x, \tag{1}$$

where P is a Hurwitz matrix, $\varphi(\sigma)$ is a differentiable function satisfying the conditions

$$\text{a)}\ \ 0 \le \sigma\varphi(\sigma) \le \mu_0\sigma^2; \qquad \text{b)}\ \ -\alpha_1 \le \varphi'(\sigma) \le \alpha_2. \tag{2}$$

Here $\mu_0, \alpha_1, \alpha_2$ are finite numbers and, without loss of generality, $\alpha_1 \ge 0$, $\alpha_2 \ge \mu_0$.** We shall also assume that the vectors $q, Pq, \ldots, P^{\nu-1}q$ are linearly independent. Introduce the transfer function of the linear part of the system $\chi(\lambda) = r^*(P - \lambda I)^{-1}q$. Let $\tau_1, \tau_2, \vartheta$ be certain parameters. Put

$$\pi(\omega)\tau_1[\mu_0^{-1} + \operatorname{Re}\chi(i\omega)] + \vartheta\operatorname{Re}[i\omega\chi(i\omega)]+$$

$$+\tau_2\omega^2[1 + (\alpha_2 - \alpha_1)\operatorname{Re}\chi(i\omega) - \alpha_1\alpha_2|\chi(i\omega)|^2]. \tag{3}$$

Theorem 1. *If, for some $\tau_1 > 0, \tau_2 \ge 0, \vartheta$ satisfying the condition*

$$+\infty \ge \varkappa_\pm = \tau_2 \lim_{\sigma\to\pm\infty} \frac{\vartheta}{\sigma^2}\left[\int_0^\sigma \varphi(s)\,ds - \frac{\sigma\varphi(\sigma)}{2}\right] \ge 0 \tag{4}$$

and for all $\omega \ge 0$ one has $\pi(\omega) > 0$, then the solution $x \equiv 0$ of system (1) is asymptotically stable in the large.

Theorem 2. *Let relation (2b) be satisfied in the strict sense, i.e.* $-\alpha_1 < \varphi'(\sigma) < \alpha_2$, *and let* $\mu_0^{-1} + \chi(i\omega) \neq 0$. *If, for some* $\tau_1 \geq 0$, $\tau_2 > 0$, ϑ, *satisfying condition (4), and all* $\omega \geq 0$, *one has* $\pi(\omega) \geq 0$, *then the solution* $x \equiv 0$ *of system (1) is asymptotically stable in the large*[**].[*]

The condition $\pi(\omega) > 0$ for $\tau_2 = 0$ is the frequency condition of V. M. Popov (2). (Then the requirement that $\varphi(\sigma)$ be differentiable and condition (2b) are absent.) Examples can be given in which the frequency condition of V. M. Popov — ⋯is not satisfied, but asymptotic stability in the large holds according to Theorems 1 and 2.

The condition $\pi(\omega) \geqslant 0$ can be given the following geometric interpretation. Denote $\zeta(\omega) = \omega^2[1 + (\alpha_2 - \alpha_1)\operatorname{Re}\chi(i\omega) - \alpha_1\alpha_2|\chi(i\omega)|^2]$, $\xi(\omega) = [\zeta(\omega)]^{-1}[\mu_0^{-1} + \operatorname{Re}\chi(i\omega)]$, $\eta(\omega) = [\zeta(\omega)]^{-1}\operatorname{Re}\{i\omega\chi(i\omega)\}$, and construct in the plane $\{\xi, \eta\}$ the curves $\Gamma_+\{\xi = \xi(\omega), \eta = \eta(\omega), \zeta(\omega) > 0\}$, $\Gamma_-\{\xi = \xi(\omega), \eta = \eta(\omega), \zeta(\omega) < 0\}$. Any straight line in the plane $\{\xi, \eta\}$ intersecting the half-axis $\eta = 0$, $\xi \leqslant 0$ will be called an admissible straight line. The closed half-plane bounded by an admissible straight line and containing the half-axis $\eta = 0$, $\xi \geqslant 0$ will be called positive; the closed half-plane bounded by an admissible straight line and not containing the half-axis $\eta = 0$, $\xi \leqslant 0$ will be called negative. The existence of parameters for which $\pi(\omega) \geqslant 0$ when $\zeta(\omega) \neq 0$ is equivalent to the existence of an admissible straight line such that the curve Γ_+ lies in the positive, and the curve Γ_- in the negative, half-planes. (In the case $\zeta(\omega) \equiv 0$, asymptotic stability in the large holds by V. M. Popov' s condition.)

Consider the system

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[*] Here and below, capital Latin letters denote $\nu \times \nu$ matrices, lowercase Latin letters denote $\nu \times 1$ column vectors, and Greek letters denote numbers. The exceptions are t, time, and V, a Lyapunov function. Indices are denoted by any letters. Unless otherwise stated, matrices, vectors, and numbers are real. The asterisk denotes Hermitian conjugation. The notation $H > 0$ ($H \geq 0$) means that $x^*Hx > 0$ for $x \neq 0$ ($x^*Hx \geq 0$); I is the identity matrix, 0 is the zero vector.

[**] The case in which one of these numbers is equal to $\pm\infty$ is treated in another way in (1). The case in which, instead of $0 \leq \sigma\varphi \leq \mu_0\sigma^2$, one has $\mu_1\sigma^2 \leq \sigma\varphi \leq \mu_2\sigma^2$ and the matrix $P + \mu_1 qr^*$ is Hurwitz, is reduced to the case considered by the substitution $\varphi = \varphi_1 + \mu_1\sigma$.

[***] Theorems 1 and 2 assert the absolute stability of the linear part of system (1) in the class of functions satisfying relations (2), and, for example, the requirement of the existence of the limits $\lim_{\sigma\to\pm\infty}\varphi(\sigma)/\sigma \leq +\infty$ (then (4) is satisfied for all ϑ).

$$dx/dt = Px + q\varphi(\sigma) + f(t, x), \qquad \sigma = r^*x, \tag{5}$$

under the previous assumptions, supposing that instead of (2a) the weaker condition

$$\lim \frac{\varphi(\sigma)}{\sigma}\left[\mu_0 - \frac{\varphi(\sigma)}{\sigma}\right] \geqslant 0 \quad \text{as } |\sigma| \to \infty$$

is fulfilled. The function $f(t, x)$ is a continuous function of t, x, and

$$\lim_{|x|\to\infty} |f(t, x)|/|x| = 0$$

uniformly in t.

Theorem 3. If, for some $\tau_1 > 0,\ \tau_2 \geqslant 0,\ \vartheta$, one has $\pi(\omega) > 0$, then system (5) is dissipative, i.e., in the space $\{x\}$ there exists a bounded closed set $\mathfrak{F}$ such that

I. From $x(t_0) \in \mathfrak{F}$ it follows that $x(t) \in \mathfrak{F}$ for $t \geqslant t_0$.

II. For every solution there is a time t_0 such that $x(t_0) \in \mathfrak{F}$.

III. There exists a solution $x_0(t) \in \mathfrak{F}$ for $-\infty < t < +\infty$.

If $f(t, x)$ is periodic in t with period χ, then there exists a periodic solution $x_0(t) \in \mathfrak{F}$ with period χ (when $\tau_2 = 0$) or with a multiple of χ (when $\tau_2 > 0$), $-\infty < t < +\infty$.

The proof of Theorems 1-3 essentially uses the following algebraic proposition.

Theorem 4. Given $\rho > 0,\ a,\ b,\ K = K^*,\ A$, the equation $\det(A - \lambda I) = 0$ has a simple zero root and the remaining roots lie in the left half-plane, $An = \bar{0},\ A^*m = \bar{0}\ (n \neq \bar{0},\ m \neq \bar{0})$, and the vectors $a, Aa, \dots, A^{\nu-1}a$ are linearly independent. With respect to the unknown matrix $H = H^*$ define

$$-G = A^*H + HA, \qquad -g = Ha + b, \qquad F = \rho(G + K) - gg^*, \tag{6}$$

and set $A_\omega = A - i\omega I,\ a_\omega = A_\omega^{-1}a,\ \pi_0(\omega) = \rho + 2\operatorname{Re} b^*a_\omega + a_\omega^*Ka_\omega$. For the existence of a matrix $H = H^*$ satisfying the quadratic inequality $F > 0$ $(F \geqslant 0)$, it is necessary and sufficient that $\pi_0(\omega) > 0$ for $\omega > 0$, $n^*Kn = \lim_{\omega\to 0}\omega^2\pi_0(\omega) > 0$ $(\pi_0(\omega) \geqslant 0)$. If $\pi_0(\omega) \geqslant 0$, then there exists a matrix $H = H^*$ satisfying the equation $F = 0$.

Below we give concise proofs of Theorems 1-4.

2°. **Proof of Theorem 4*. Necessity.** We have

$$a_\omega^*Ga_\omega = -a_\omega^*(A_\omega^*H + HA_\omega)a_\omega = 2\operatorname{Re} a_\omega^*(g + b),$$

therefore from the last relation (6) it follows that

$$a_\omega^*Fa_\omega = \rho\pi_0(\omega) - |g^*a_\omega - \rho|^2, \tag{7}$$

i.e., $\pi_0(\omega) > 0$ ($\geqslant 0$) for $F > 0$ ($\geqslant 0$). Since $n^*Gn = 0$, from (6) we have $\rho n^*Kn = n^*Fn + (n^*g)^2 > 0$ ($\geqslant 0$) for $F > 0$ ($\geqslant 0$).

Sufficiency. The case $F > 0$. As $\omega \to 0$ we have

$$\pi_0(\omega) = -\omega^{-2}(m^*a)^2(m^*n)^{-2}n^*Kn + O(1),$$

$$a_\omega^* F a_\omega = \omega^{-2}(m^*a)^2(m^*n)^{-2}n^*Fn + O(1).$$

Therefore there exists $F > 0$ such that $a_\omega^* F a_\omega < \pi_0(\omega)$ for $0 < \omega < \infty$. As in (4) (see also (^{2}), p. 131), it is easy to show that there exists a vector g satisfying identity (7). From the asymptotics as $\omega \to 0$ it follows that $n^*Gn = 0$, where G is determined by the last equality (6). Therefore there exists a solution $H_0 = H_0^*$ of the first equation (6), and the general solution of (6) is

$$H = H_0 + \xi m m^*$$

with arbitrary ξ. Let $h = Ha + b + g$. From (7) it follows that $\operatorname{Re} a_\omega^* h = 0$. Let

$$h^*(A - \lambda I)^{-1}a = \psi(\lambda) + \gamma/\lambda,$$

where $\psi(\lambda)$ is holomorphic in the right closed half-plane. From the condition $\operatorname{Re}\psi(i\omega) \equiv 0$ it follows, by the principle of analytic continuation, that $\psi(\lambda) \equiv$ const, i.e. $\psi(\lambda) \equiv 0$. Since

$$\gamma = h^*n \cdot m^*a \cdot (m^*n)^{-1},$$

one can choose ξ so that $\gamma = 0$. Then $h^*(A - \lambda I)^{-1}a \equiv 0$, $h = 0$, i.e. the second relation (6) is fulfilled.

The case $F \geqslant 0$ is considered similarly, with some simplifications.

3°. **Proof of Theorems 1 and 2.** It is enough to consider the case $\tau_2 > 0$. Denote

$$y = \begin{pmatrix} x \\ \varphi \end{pmatrix}, \qquad a = \begin{pmatrix} \bar{0} \\ 1 \end{pmatrix}, \qquad c = \begin{pmatrix} P^*r \\ q^*r \end{pmatrix}, \qquad r_1 = \begin{pmatrix} r \\ 0 \end{pmatrix}, \qquad A = \begin{pmatrix} P & q \\ \bar{0}^* & 0 \end{pmatrix}.$$

Taking as Lyapunov function *

$$V(x) = y^*Hy + \vartheta \int_0^\sigma \varphi \, d\sigma,$$

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* We note that an analogous theorem for the case when A is a Hurwitz matrix is simply derived from $^{(3)}$ for $F > 0$ and from $^{(4)}$, $F \geqslant 0$. Theorem 4 can be obtained from $^{(3)}$ by the corresponding limiting passage. The direct proof given below is simpler; the proof of necessity repeats $^{(3)}$, while in the proof of sufficiency Kalman' s idea $^{(4)}$ is used.

we obtain **

$$-\dot{V} = [y^*(G+K)y + 2y^*g\dot{\varphi} + \tau_2\dot{\varphi}^2] +$$
$$+\tau_1(\mu_0\sigma - \varphi)\varphi + \tau_2(\varphi' + \alpha_1\sigma)(\alpha_2 - \varphi')\dot{\sigma}^2, \tag{8}$$

where

$$y^*Ky = \vartheta y^*c \cdot a^*y + \tau_1 a^*y \cdot (a^*y - \mu_0 r_1^* y) - \tau_2\alpha_1\alpha_2(c^*y)^2;$$

$K = K^*$; G, g have the form (6);

$$b = -\frac{\tau_2}{2}(\alpha_2 - \alpha_1)c.$$

Using the expressions for a, b, K and taking $\rho = \tau_2$, after simple transformations we obtain that the function $\pi_0(\omega)$ in Theorem 4 has the form $\pi_0(\omega) = \omega^{-2}\pi(\omega)$ (see (3)). Therefore, by Theorem 4, there exists a matrix $H = H^*$ such that the expression in square brackets in (8) is a positive definite (Theorem 1) or nonnegative (Theorem 2) form in y and $\dot{\varphi}$, i.e. $\dot{V} < 0$ for $x \neq 0$ (Theorem 1) or $\dot{V} \leqslant 0$ (Theorem 2). To prove Theorem 1 it remains, according to Theorem 4 (^{6}), to show the absence of solutions for which $|x(t)| \to \infty$. Arguing as in (7) (Appendix, § 2), we obtain that ***

$$\Omega_0(x) \equiv y^*Hy + \vartheta\sigma\varphi/2 \geqslant \varepsilon_0|x|^2$$

for some $\varepsilon_0 > 0$. If $|x(t)| \to \infty$, on the one hand,

$$\dot{V} \to -\infty, \qquad \dot{V} < 0, \qquad V \to -\infty,$$

and, on the other hand, from (4) it follows that

$$V = \Omega_0(x) + \vartheta\left[\int_0^\sigma \varphi\, d\sigma - \frac{\sigma\varphi}{2}\right] \to +\infty.$$

The contradiction proves Theorem 1.

To prove Theorem 2 we apply Lemma 5 (^{1}) (Appendix, § 3). From $\dot{V} \equiv 0$ it follows, according to (8), that $\dot{\sigma} \equiv 0$, $\sigma \equiv \sigma_0 = \text{const}$. From the ident–

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[*] From the results of (^{5}) it follows that, irrespective of the use of additional information about the nonlinearity, Popov' s frequency condition cannot be improved in the class of Lyapunov functions

$$V = x^*H_1x + \vartheta\int \varphi\, d\sigma$$

(H_1 and ϑ are parameters).

[**] To the expression obtained after differentiation there has been added and subtracted the expression, nonnegative by virtue of (2),

$$\tau_1(\mu_0\sigma - \varphi)\varphi + \tau_2(\varphi' + \alpha_1\sigma)(\alpha_2 - \varphi')\dot{\sigma}^2.$$

*** The function $\Omega_0(x)$ is obtained from V by replacing $\varphi = \mu\sigma$, $\mu = \text{const}$, $0 \leqslant \mu \leqslant \mu_0$, and, after integration, replacing $\mu = \varphi/\sigma$.

from the equality
$\sigma_0 = r^*[\exp(Pt)x_0 - Pq^{-1}\varphi(\sigma_0)]$ we obtain that
$f^* \exp(Pt)x_0 = 0$, $\sigma_0 + \chi(0)\varphi(\sigma_0) = 0$, which, for $\varphi(\sigma_0) \neq 0$, contradicts assumptions (2a), $\pi(0) \geq 0$, $\chi(0) \neq \mu_0^{-1}$. Consequently, $\varphi(\sigma_0) = 0$, $x(t) \to 0$. Thus, conditions (II), (III) of Lemma 5 (^{1}) are satisfied. From the inequalities $\pi(\omega) \geq 0$, $\chi(i\omega) \neq \mu_0^{-1}$, it is easy to derive that the curve $\chi(i\omega)$ does not intersect the segment $[-\mu_0^{-1}, -\infty)$, i.e., the matrix $P + \mu qr^*$ is Hurwitz for $0 \leq \mu \leq \mu_0$, whence follows the fulfillment of condition (I) of Lemma 5 (^{1}) and the inequality $\Omega_0(x) \geq \varepsilon_0|x|^2$. Hence, as above, we obtain the fulfillment of the last condition (IV$'$) of Lemma 5 (^{1}).

4°. The proof of Theorem 3 is carried out according to the same scheme as the proof of Theorem 4 (^{7}) (which is obtained from Theorem 3 for $\tau_2 = 0$ and in the absence of condition (26)); moreover, as the Lyapunov function one should take the function constructed above,

$$V = y^*Hy + \vartheta \int_0^\sigma \varphi \, d\sigma.$$

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