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Abstract

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Isoscalar Tensor Operators of the Group SU_3 and Relations Between Particle Masses in the Octet Model

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1. In any symmetric model of elementary particles, irreducible tensor operators prove useful, i.e., operators transforming according to an irreducible representation of the symmetry group. In particular, they are convenient for investigating the mass spectrum and scattering amplitudes. An example of their application is Okubo' s mass formula in a model based on the group SU_3.

Any matrix element of a tensor operator can be expressed in terms of the Clebsch –Gordan coefficients of the group (see ([1])). If nothing is known about the operator except its transformation properties, then arbitrary parameters enter such an expression. Eliminating them, one can obtain relations between matrix elements that are consequences of the transformation properties of the operator. Below another method is described, making it possible to obtain such relations directly and in explicit form.

The method is applied to tensor operators of a special kind, which occur in the SU_3-model of strong interactions (isoscalar and hyperscalar operators). Relations between particle masses in the Okubo approximation are obtained for an arbitrary representation. They turn out to be a simple generalization of the relations that exist in the unitary octet and decuplets.

2. Consider the group SU_3 and denote its irreducible representations by $\mu = (p, q)$, and the basis vectors of these representations by $|\mu\nu\rangle$, where $\nu = (ytt_3)$ –hypercharge, isospin, and the projection of isospin ([1]). Then to each element of the group there corresponds the transformation

$$u(|\mu\nu\rangle) = \sum_{\nu'} |\mu\nu'\rangle \overset{*}{D}{}^{(\mu)}_{\nu'\nu} = U|\mu\nu\rangle,$$

$$u(\langle\mu\nu|) = \sum_{\nu'} \langle\mu\nu'|D^{(\mu)}_{\nu'\nu} = \langle\mu\nu|U^{-1}, \qquad (1)$$

where $\mathcal{D}^{(\mu)}_{\nu'\nu} = D^{(\mu)}_{\nu'\nu}(u)$ are the matrices realizing the representation μ. Let the operator T^μ_ν act in the space spanned by the vectors $|\mu'\nu'\rangle$ (with all possible μ' and ν'). Then it can be represented in the form

$$T^\mu_\nu = \sum_{\mu_1\nu_1\mu_2\nu_2} |\mu_1\nu_1\rangle\langle\mu_2\nu_2| \cdot \langle\mu_1\nu_1|T^\mu_\nu|\mu_2\nu_2\rangle. \tag{2}$$

Under the transformation (1) it goes over into

$$u(T^\mu_\nu) = \sum_{\mu_1\nu_1\mu_2\nu_2} U|\mu_1\nu_1\rangle\langle\mu_2\nu_2|U^{-1} \cdot \langle\mu_1\nu_1|T^\mu_\nu|\mu_2\nu_2\rangle = UT^\mu_\nu U^{-1}. \tag{3}$$

An operator T^μ_ν is called an irreducible tensor operator if it transforms as the vector $|\mu\nu\rangle$:

$$u\,(T^\mu_\nu) = \sum_{\nu'} T^\mu_{\nu'} D^{*(\mu)}_{\nu'\nu}. \tag{4}$$

Comparing (4), (3), and (1), we obtain

$$\sum_{\nu_1\nu_2} D^{*(\mu_1)}_{\nu_1'\nu_1} D^{(\mu_2)}_{\nu_2'\nu_2} \langle\mu_1\nu_1|T^\mu_\nu|\mu_2\nu_2\rangle = \sum_{\nu'} \langle\mu_1\nu_1'|T^\mu_{\nu'}|\mu_2\nu_2'\rangle D^{*(\mu)}_{\nu'\nu}. \tag{5}$$

Equality (5) makes it easy to prove the following theorem.

Theorem 1. Let $\Phi(\mu\nu)$ be vectors transforming as $|\mu\nu\rangle$. Then, in order that the numbers $\langle\mu_1\nu_1|T^\mu_\nu|\mu_2\nu_2\rangle$ be matrix elements of some tensor operator T^μ_ν, it is necessary and sufficient that the vectors

$$\Phi^\mu_\nu(\mu_1\mu_2) = \sum_{\nu_1\nu_2} \Phi(\mu_1\nu_1)\Phi^*(\mu_2\nu_2)\langle\mu_1\nu_1|T^\mu_\nu|\mu_2\nu_2\rangle \tag{6}$$

transform as $|\mu\nu\rangle$.

The idea of the proposed method is as follows. Take a complete set of commuting operators consisting of two independent invariants of the group SU_3 (denote them by F^2 and E^3) and the operators Y, T^2, and T_3 (hypercharge, the square of isospin, and the projection of isospin), and apply each of these operators to the vector Φ^μ_ν. As a consequence of Theorem 1, Φ^μ_ν is an eigenvector of the listed operators with eigenvalues respectively $F^2(\mu)$, $E^3(\mu)$, y, $t(t+1)$, and t_3, where $(ytt_3) = \nu$. On the other hand, the action of these operators on the right-hand side of equality (6) can be found by knowing the action of the generators of the group on the vectors $\Phi(\mu\nu)$. Equating the expressions found by these two methods, we obtain the desired relations between the matrix elements $\langle\mu_1\nu_1|T^\mu_\nu|\mu_2\nu_2\rangle$.

3. The method described was applied in the particular case where $\nu = 0 = (000)$. A vector with such quantum numbers is contained only in representations of the form $\sigma = (s, s)$. Thus, the operator considered was $T_0^\sigma = T_{000}^{(s,s)} = T^s$. Such an operator is invariant with respect to the subgroup of isotopic rotations and the hypercharge gauge group. Therefore it may be called an isoscalar and hypercharge-scalar tensor operator.

Let us note that, owing to the special choice of the representation $\sigma = (s, s)$, only one of the two invariants of the group is sufficient for fixing the transformation properties. As this invariant we used the operator $F^2 = F_a^{aF}$, quadratic in the generators F^a of the group (1). In order to find the result of its action on the right-hand side of (6), the explicit form of the matrix elements of the generators (1) was used. Instead of checking the action on Φ_ν^μ of the operators Y, T^2, T_3, the necessary transformation properties were ensured with the aid of the Clebsch–Gordan coefficients of the subgroup SU_2. The result of the calculations can be stated in the form of the following theorem.

Theorem 2. In order that the quantities $\langle \mu_1 y^{(1)} t^{(1)} t_3^{(1)} | T^s | \mu_2 y^{(2)} t^{(2)} t_3^{(2)} \rangle$ be matrix elements of some tensor operator $T^s = T_{000}^{(s,s)}$ of the group SU_3, it is necessary and sufficient that they:

1) be different from zero only when

$$y^{(1)} = y^{(2)} = y, \qquad t^{(1)} = t^{(2)} = t, \qquad t_3^{(1)} = t_3^{(2)} = t_3; \tag{7}$$

2) be independent of t_3:

$$\langle \mu_1 y t t_3 | T^s | \mu_2 y t t_3 \rangle = a_s^{\mu_1 \mu_2}(yt); \tag{8}$$

3) satisfy the relations

$$\begin{gathered}
\left[{}^3/_2 y^2 + 2t(t+1) + \Lambda^{s\mu_1\mu_2}\right] a^{s\mu_1\mu_2}(y,t) + \\
+ \frac{1}{2t+1}\{A^{\mu_1\mu_2}({}^1/_2 y + t) a^{s\mu_1\mu_2}(y-1, t - {}^1/_2) + \\
+ A^{\mu_1\mu_2}({}^1/_2 y + t + 1) a^{s\mu_1\mu_2}(y+1, t + {}^1/_2) + \\
+ A^{\mu_1\mu_2}({}^1/_2 y - t) a^{s\mu_1\mu_2}(y+1, t - {}^1/_2) + \\
+ A^{\mu_1\mu_2}({}^1/_2 y - t - 1) a^{s\mu_1\mu_2}(y-1, t + {}^1/_2)\} = 0,
\end{gathered} \tag{9}$$

where

$$\begin{gathered}
\Lambda^{s\mu_1\mu_2} = F^2(s) - F^2(\mu_1) - F^2(\mu_2), \\
A^{\mu_1\mu_2}(I) = \sqrt{A^{\mu_1}(I) A^{\mu_2}(I)}, \\
A^\mu(I) = E^3(\mu) + [F^2(\mu) + 1] I^2 - I^3, \\
F^2(\mu) = F^2(p,q) = {}^1/_3[(p+q)^2 - pq] + p + q, \\
E^3(\mu) = E^3(p,q) = {}^1/_3(p-q)\{{}^1/_9[2(p+q)^2 + pq] + p + q + 1\},
\end{gathered} \tag{10}$$

y, t are such that the vectors $|\mu_1 ytt_3\rangle$ and $|\mu_2 ytt_3\rangle$, $-t \leq t_3 \leq t$, exist simultaneously.

In formulas (10), E^3 denotes the second independent invariant of the group SU_3, cubic in the generators.

4. Tensor operators of type T^s are used to find the spectrum of particle masses in the octet model of strong interactions. For example, in the Okubo approximation the mass operator has the form ([1])

$$M = T^0 + T^1. \tag{11}$$

Such an operator could be studied by means of formula (9). However, much more transparent results are obtained if, for this particular case, all the calculations are carried out anew, in a somewhat different way.

Let us denote the matrix elements of the operator M by

$$\langle \mu_1 ytt_3 | M | \mu_2 ytt_3 \rangle = m^{\mu_1 \mu_2}(y, t). \tag{12}$$

Arguments analogous to those carried out in item 2 show that the numbers $m^{\mu_1\mu_2}(y,t)$ are the matrix elements of some operator of the form (11) if and only if the vector

$$\Phi^{01}(\mu_1 \mu_2) = \sum_{ytt_3} \Phi(\mu_1 ytt_3) \Phi^*(\mu_2 ytt_3) m^{\mu_1 \mu_2}(y, t) \tag{13}$$

transforms as a vector of the (reducible) representation $\sigma_{01} = (0,0) \oplus (1,1)$ with quantum numbers $v = 0 = (000)$. The zero quantum numbers of the vector Φ^{01} are in fact already guaranteed by the fact that the matrix elements are chosen independent of t_3 (12). In order to fix the representation to which the vector belongs, we used earlier the invariant F^2. Now let us proceed differently. Among the generators of the group SU_3 there is a generator K_+ (see (1)) which increases the hypercharge by one. Let us act on the vector Φ^{01} with the square of this operator. Then, taking into account that in the representation σ_{01} the hypercharge $y \leq 1$, we obtain

$$(K_+)^2 \Phi^{01} = 0. \tag{14}$$

Conversely, if the vector Φ^{01}, having zero quantum numbers, satisfies (14), then it belongs to the representation σ_{01}. The result of the action of the operator $(K_+)^2$ on the right-hand side of (13) can be calculated by using the explicit form of the operator K_+ (1). Equating this expression to zero, we obtain the following relations (for simplicity we omit the indices $\mu_1\mu_2$ on $m^{\mu_1\mu_2}(y,t)$):

$$A_1^{11}(yt)m(y-1,t-\tfrac{1}{2})+A_1^{22}(yt)m(y+1,t+\tfrac{1}{2})-$$

$$-2A_1^{21}(yt)m(y,t)=0,$$

$$A_2^{11}(yt)m(y-1,t+\tfrac{1}{2})+A_2^{22}(yt)m(y+1,t-\tfrac{1}{2})-$$

$$-2A_2^{12}(yt)m(y,t)=0, \tag{15}$$

$$2(t+1)[A_3^{11}(yt)m(y-1,t+\tfrac{1}{2})+A_3^{22}(yt)m(y+1,t+\tfrac{1}{2})]-$$

$$-(2t+3)A_3^{12}(yt)m(y,t)-(2t+1)A_3^{21}(yt)m(y,t+1)=0,$$

where

$$A_1^{ij}(yt)=\sqrt{A^{\mu_i}(\tfrac{1}{2}y+t)\,A^{\mu_j}(\tfrac{1}{2}y+t+1)},$$

$$A_2^{ij}(yt)=\sqrt{A^{\mu_i}(\tfrac{1}{2}y-t)\,A^{\mu_j}(\tfrac{1}{2}y-t-1)}, \tag{16}$$

$$A_3^{ij}(yt)=\sqrt{-A^{\mu_i}(\tfrac{1}{2}y+t+1)\,A^{\mu_j}(\tfrac{1}{2}y-t-1)}.$$

In the case where no mixing of unitary multiplets occurs, the mass operator M has no off-diagonal matrix elements (with $\mu_1\neq\mu_2$), while the diagonal elements $m(yt)=m^{\mu\mu}(yt)$ are equal to the masses (or squares of masses) of the corresponding particles. In this case the equalities (15) give directly relations between particle masses. Since $\mu_1=\mu_2=\mu$, in each of these equalities one may take outside the brackets a factor of the form $A_k^{11}(yt)$. Analysis shows that these factors may be canceled. As a result we obtain:

Theorem 3. In order that the numbers $m(yt)=m^{\mu\mu}(yt)$ be matrix elements of a diagonal operator of the form (11), it is necessary and sufficient that the following relations be satisfied:

1) for any three vectors $|\mu ytt_3\rangle$, $|\mu,y+1,t+\frac{1}{2},t_3'\rangle$, $|\mu,y-1,t-\frac{1}{2},t_3''\rangle$:

$$m(y,t)=\tfrac{1}{2}[m(y+1,t+\tfrac{1}{2})+m(y-1,t-\tfrac{1}{2})]; \tag{17}$$

2) for any three vectors $|\mu ytt_3\rangle$, $|\mu,y+1,t-\frac{1}{2},t_3'\rangle$, $|\mu,y-1,t+\frac{1}{2},t_3''\rangle$:

$$m(y,t) = \tfrac{1}{2}\big[m(y+1, t-\tfrac{1}{2}) + m(y-1, t+\tfrac{1}{2})\big]; \tag{18}$$

3) for any four vectors $|\mu y t t_3\rangle$, $|\mu, y, t+1, t_3'\rangle$, $|\mu, y+1, t+\frac{1}{2}, t_3''\rangle$, $|\mu, y-1, t+\frac{1}{2}, t_3'''\rangle$:

$$(t+1)\big[m(y+1, t+\tfrac{1}{2}) + m(y-1, t+\tfrac{1}{2})\big] =$$

$$= \tfrac{1}{2}\big[(2t+3)m(y,t) + (2t+1)m(y,t+1)\big]. \tag{19}$$

If M is the mass operator, then the first two relations mean the equidistance of masses (or squares of masses) along the straight lines $\frac{1}{2}y \pm t = \text{const}$, which generalizes the equidistance relations in decuplets (representations $(3,0)$ and $(0,3)$). The third relation simply generalizes the mass relation in the unitary octet $(1,1)$.

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1. J. J. de Swart, Rev. Mod. Phys., **35**, No. 4, 916 (1963).

Note: Figure translations are in progress. See original paper for figures.