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# MATHEMATICS

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**Abstract**

**Full Text**

MATHEMATICS

B. D. ROMM

## AN ANALOGUE OF THE PLANCHEREL FORMULA FOR THE REAL UNIMODULAR GROUP OF THE THIRD ORDER

*(Presented by Academician I. M. Vinogradov on 7 IX 1964)*

For the first time, the problem of decomposing the regular representation into irreducible representations for the real unimodular group was solved in papers <sup>(1, 2)</sup> for the case of the second order. In paper <sup>(2)</sup> the Plancherel formula was obtained. In the present note, for simplicity and brevity of exposition, the Plancherel formula is given for the group of the third order  $G_3$ . However, the method used by the author extends to the real unimodular group of any order.

The irreducible representations of the principal series were constructed in paper <sup>(3)</sup> for the group of any order. We briefly describe these representations in the case of the group  $G_3$ .

Let  $K$  denote the subgroup in the group  $G_3$  consisting of all matrices  $k = \|a_{ij}\|$ ,  $i, j = 1, 2, 3$ , where  $a_{31} = a_{32} = 0$ , and let  $X$  be the subgroup for whose matrices  $x = \|\xi_{ij}\|$  one has  $\xi_{11} = \xi_{22} = \xi_{33} = 1$ ,  $\xi_{21} = 0$ ,  $\xi_{ij} = 0$  for  $i < j$ . For every  $g \in G_3$  and almost all  $x \in X$  there is a decomposition  $xg = k_1x_1$  with

$$k_1 \in K, \quad x_1 \in X.$$

Let  $k_1 = \|a'_{ij}\|$ ,  $z_1 = \frac{a'_{11}z + a'_{21}}{a'_{12}z + a'_{22}}$ . Denote

$$\dot{z} = \begin{pmatrix} 1 & 0 & 0 \\ z & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{Im } z \neq 0.$$

$\dot{z}_1$  denotes the same as  $\dot{z}$ , except that  $z$  is replaced by  $z_1$ ;  $\Lambda = a'_{11}a'_{22} - a'_{12}a'_{21}$ ;  $\beta(K)$  is the ratio of the left-invariant measure on the group  $K$  to the right-invariant one, normalized so that  $\beta(e) = 1$ , where  $e$  is the identity in the group  $G_3$ .

Consider the totality of all functions  $f(\dot{z}x)$  on the manifold of products  $\dot{z}x$ , analytic in  $z$  for  $\text{Im } z \neq 0$  and measurable with respect to the parameters  $\xi_{31}$  and  $\xi_{32}$  of the matrix  $x$ , for which the integral

$$\int |f(\dot{z}x)|^2 |\operatorname{Im} z|^{n-2} d\xi_{31} d\xi_{32} dz, \quad dz = dx dy, \quad z = x + iy, \quad n = 2, 3, 4, \dots,$$

converges for some fixed  $n$ . Let  $\rho \in (-\infty, +\infty)$ ,  $\varepsilon = 0, 1$ ; then the operator

$$T_g f(\dot{z}x) = (a'_{12}z + a'_{22})^{-n} |\Lambda|^{i\rho+n/2} (\operatorname{sgn} \Lambda)^\varepsilon \beta^{-1/2}(K_1) f(\dot{z}_1 x_1) \quad (1)$$

defines a unitary representation, irreducible in the Hilbert space  $\mathcal{H}'_n$  of functions  $f(\dot{z}x)$  of the indicated type, where the scalar product is determined by the formula

$$(f, f_1) = \int f(\dot{z}x) \overline{f_1(\dot{z}x)} |\operatorname{Im} z|^{n-2} d\xi_{31} d\xi_{32}.$$

In paper <sup>(3)</sup>  $\varepsilon = 0$ . This series of irreducible representations is denoted there by  $d_1$ .

Let us construct one more series of irreducible unitary representations of the group  $G_3$ . Let  $K_0$  denote the subgroup of all matrices  $k_0 = \|v_{ij}\|$  in  $G_3$ , where  $v_{ij} = 0$  if  $i > j$ ;  $X_0$ , the subgroup of all matrices  $x = \|\theta_{ij}\|$ , where  $\theta_{ij} = 0$  for  $i < j$ ,  $\theta_{11} = \theta_{22} = \theta_{33} = 1$ . By analogy with the preceding case, for almost all  $x_0 \in X_0$  and for each fixed  $g \in G_3$  we have  $x_0 g = k'_0 x'_0$ , with  $k'_0 \in K_0$ ,  $x'_0 \in X_0$ ;  $\beta_0(K_0)$  is the ratio of the left-invariant measure on the group  $K_0$  to the right-invariant one, with  $\beta_0(e) = 1$ ;  $\varepsilon_1, \varepsilon_2 = 0, 1$ ;  $\rho_1, \rho_2 \in (\infty, +\infty)$ ,  $k'_0 = \|v'_{ij}\|$ . The operator

$$T_g f(x_0) = |v'_{11}|^{i\rho_1} (\operatorname{sgn} v'_{11})^{\varepsilon_1} |v'_{22}|^{i\rho_2} (\operatorname{sgn} v'_{22})^{\varepsilon_2} \beta^{-1/2}(k'_0) f(x'_0) \quad (2)$$

in the Hilbert space  $L_2(X_0)$  of functions  $f(x_0)$  measurable on the group  $X_0$  and with summable square of the modulus with respect to the invariant measure on  $X_0$ , defines an irreducible unitary representation of the group  $G_3$ . The totality of all these representations in (3) is denoted by  $d_0$ .

The series  $d_0$  and  $d_1$  exhaust all representations that enter into the decomposition of the regular representation into irreducibles.

Let us indicate the normalization of the invariant measure in the Plancherel formula. If  $d$  denotes the sign of differentiation, then the exterior product of the differential forms  $g_{ij}$ , where  $(i, j) \neq (3, 3)$ ,  $dg \cdot g^{-1} = \|g_{ij}\|$ ,  $g \in G_3$ , forms an invariant measure  $d\mu(g)$  on  $G_3$ .

As is known, for a finite infinitely differentiable function  $x(g)$  on the group  $G_3$  and an irreducible unitary representation  $g \mapsto T_g$ , the operator

$$T_x = \int x(g)T_g d\mu(g)$$

has a trace in the corresponding representation space. When  $T_g$  is defined by formula (1), we shall denote this trace by  $\text{Sp}(T_{x,n,\rho,\varepsilon})$ , and in the case (2), by  $\text{Sp}(T_{x,\rho_1,\rho_2,\varepsilon_1,\varepsilon_2})$ .

The Plancherel formula has the form

$$\begin{aligned} x(e) = \frac{1}{2^8 \pi^5} \left\{ \sum_{k=2}^{\infty} (k-1) \int [(k-1)^2 + 4\rho^2] [\text{Sp}(T_{x,k,\rho,0}) + \text{Sp}(T_{x,k,\rho,1})] d\rho + \right. \\ + 2 \iint_{\rho_1 > \rho_2 > 0} \rho_1 \rho_2 (\rho_1 - \rho_2) \left[ \text{th} \frac{\pi(\rho_1 - \rho_2)}{2} + \text{th} \frac{\pi\rho_2}{2} - \text{th} \frac{\pi\rho_1}{2} \right] \times \\ \times \text{Sp}(T_{x,\rho_1,\rho_2,0,0}) d\rho_1 d\rho_2 + 2 \iint_{\rho_1 > \rho_2 > 0} \rho_1 \rho_2 (\rho_1 - \rho_2) \left[ \text{cth} \frac{\pi(\rho_1 - \rho_2)}{2} + \right. \\ \left. + \text{th} \frac{\pi\rho_2}{2} - \text{cth} \frac{\pi\rho_1}{2} \right] \text{Sp}(T_{x,\rho_1,\rho_2,1,0}) d\rho_1 d\rho_2 + 2 \iint_{\rho_1 > \rho_2 > 0} \rho_1 \rho_2 (\rho_1 - \rho_2) \times \\ \times \left[ \text{cth} \frac{\pi(\rho_1 - \rho_2)}{2} + \text{cth} \frac{\pi\rho_2}{2} - \text{th} \frac{\pi\rho_1}{2} \right] \text{Sp}(T_{x,\rho_1,\rho_2,0,1}) d\rho_1 d\rho_2 + \\ \left. + 2 \iint_{\rho_1 > \rho_2 > 0} \rho_1 \rho_2 (\rho_1 - \rho_2) \left[ \text{th} \frac{\pi(\rho_1 - \rho_2)}{2} + \text{cth} \frac{\pi\rho_2}{2} - \text{cth} \frac{\pi\rho_1}{2} \right] \times \right. \\ \left. \times \text{Sp}(T_{x,\rho_1,\rho_2,1,1}) d\rho_1 d\rho_2 \right\}. \end{aligned}$$

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*Note: Figure translations are in progress. See original paper for figures.*

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