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Abstract

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ON SOME PROPERTIES OF SOLUTIONS OF NONLINEAR EQUATIONS OF PARABOLIC TYPE IN HILBERT SPACE

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We shall say that the linear unbounded operators $A(t)$ $(0 \leq t < \infty)$, acting in the real Hilbert space H and having an everywhere dense domain of definition $D(A)$ independent of t, satisfy condition (A) if they are self-adjoint, positive definite, the strong derivative $A'(t)$ exists, and

$$(A'(t)x, x) \leq a(A(t)x, x) \qquad (x \in D(A)),$$

where $a(t)$ is a certain function defined on $[0, \infty)$. Further, we shall say that the unbounded operators $B(t, \dot{x})$ $(0 \leq t < \infty)$, acting in H with domain of definition $D(B)$, satisfy condition (B_0), if

$$(B(t, \dot{x}), \dot{x}) \geq 0 \qquad (\dot{x} \in D(B)).$$

1. Boundedness of solutions. Let $F(t, x)$ $(0 \leq t < \infty,\ x \in D(A^{1/2}))$ be a family of nonlinear functionals differentiable in the sense of Gâteaux,

$$\lim_{\lambda \to 0} \frac{F(t, x + \lambda h) - F(t, x)}{\lambda} = (P(t, x), h).$$

Consider equations with a nonlinear potential operator

$$B[t, \dot{x}(t)] + A(t)x(t) + P[t, x(t)] = 0. \tag{1,1}$$

Theorem 1. Let the operators $A(t)$ and $B(t, x)$ satisfy conditions (A) and (B_0), respectively. Let $F_t'(t, x)$ exist and

$$F_t'(t, x) \leq a(t)F(t, x) \qquad (0 \leq t < \infty,\ x \in D(A^{1/2})).$$

Then for any solution $x(t)$ of equation $(1, 1)$, with $x(0) \in D(A^{1/2}(0))$ and $t \geq 0$, the estimate holds

$$\|A^{1/2}(t)x(t)\|^2 + 2F[t, x(t)] \leq \{\|A^{1/2}(0)x(0)\|^2 + 2F[0, x(0)]\} \exp\left[\int_0^t a(s)\, ds\right]. \tag{1,2}$$

1.1. Let $F(t, x) \geq 0$. Then from $(1, 2)$, for $t \geq 0$ we have

$$\|A^{1/2}(t)x(t)\| \leq \{\|A^{1/2}(0)x(0)\| + \sqrt{2F[0, x(0)]}\} \exp\left[2\int_0^t a(s)\, ds\right]. \tag{1,3}$$

It follows from $(1, 3)$ that if

$$\int_0^\infty a(s)\, ds < \infty, \tag{1,4}$$

then $\|A^{1/2}(t)x(t)\|$ is bounded, and the zero solution of equation $(1, 1)$ is stable, i.e., for sufficiently small $\|A^{1/2}(0)x(0)\|$ and $F[0, x(0)]$ sufficiently small;

$\|A^{1/2}(t)x(t)\|$, and if it is assumed that

$$\int_0^\infty \alpha(s)\, ds = -\infty, \tag{1.5}$$

then $\lim_{t\to\infty} \|A^{1/2}(t)x(t)\| = 0$, i.e. the zero solution of equation (1.1) is asymptotically stable.

1.2. Let $F(t, x) \geqslant 0$, $\|A^{1/2}(0)x(0)\| + \sqrt{2F[0, x(0)]} \leqslant C$, and let (1.4) be fulfilled; then from (1.3) it follows that

$$\|A^{1/2}(t)x(t)\| \leqslant r \qquad (t \geqslant 0). \tag{1.6}$$

Let B be the identity operator and consider the problem

$$\dot{x} + A(t)x + P(t, x) = 0, \qquad x(0) = x_0. \tag{1.7}$$

In [1] (see also [2]) sufficient conditions were found for the local solvability of problem (1.7). From that same work it is clear that, in order to obtain a nonlocal existence theorem, it suffices to have an a priori estimate for $\|A^{1/2}(t)x(t)\|$. Thus, if one assumes that the potential operator $P(t, x)$ satisfies some Hölder condition in the aggregate of the variables, i.e. satisfies sufficient conditions for the local solvability of problem (1.7) (see [1]), then from the estimate (1.6) proved by us there follows the nonlocal solvability of problem (1.7).

1.3. Let $H = L^2(G)$, where G is a bounded domain of n-dimensional Euclidean space with smooth boundary Γ, and

$$A(t)x \equiv -\sum_{i,j=1}^{n} \frac{\partial\left[a_{ij}(t,s)x_{s_j}\right]}{\partial s_i} + ax, \qquad x|_\Gamma = 0, \tag{1.8}$$

where $a_{ij} = a_{ji}$, $\sum a_{ij}\xi_i\xi_j \geqslant 0$. We shall assume that a_{ij}, a, $\partial a_{ij}/\partial s_i$ have derivatives with respect to t and that they are bounded above by the function $\alpha(t)$ on $[0,\infty)$. Then the operator-function $A(t)$, defined by formula (1.8) on the set $\dot{W}_2^2(G)$, satisfies condition (A) (see [3]). A trivial example of an operator B satisfying condition (B_0) is the identity operator. An example of a more general operator satisfying condition (B_0) is the operator

$$B(t,\dot{x}) \equiv -\sum_{i,j=1}^{n} \frac{\partial\left[b_{ij}(t,s)\dot{x}_{s_j}\right]}{\partial s_i} + g(t,\dot{x}), \qquad x|_\Gamma = 0, \tag{1.9}$$

if it is assumed that

$$\sum_{i,j=1}^{n} b_{ij}\xi_i\xi_j \geqslant 0, \qquad g(t,\dot{x})\dot{x} \geqslant 0. \tag{1.10}$$

Now let us give an example of a functional $F(t,x)$ in $L^2(G)$. Let $\Phi(t,x)$ be a continuous scalar function, positive for $x > 0$ and $t \in [0,\infty)$; moreover,

$$\Phi_t'(t,x) \leqslant \alpha(t)\Phi(t,x), \qquad |\Phi_x'(t,x)| \leqslant Mx^m, \tag{1.11}$$

where $m = 1/(n-2)$, if $n > 2$; $m < \infty$, if $n \leqslant 2$. Then the functional

$$F(t,x) \equiv \frac{1}{2}\int_G \Phi[t, x^2(t,s)]\,ds \geqslant 0$$

satisfies the conditions of Theorem 1, and $P(t,x) = \Phi_x'(t,x^2)x$.

Theorem 2. Let a_{ij}, a, $\partial a_{ij}/\partial s_i$ be differentiable with respect to t and let these derivatives be bounded by the function $\alpha(t)$. Let the functions b_{ij} and $g(t,\dot{x})$ satisfy condition (1.10). Further, let the scalar function $\Phi(t,x)$ be positive for $x > 0$ and let (1.11) be fulfilled.

Then, if (1.4) is satisfied, the solutions of the problem

$$B[t, \partial x(t,s)/\partial t] + A(t)x(t,s) + x(t,s)\Phi_x'[t,x^2(t,s)] = 0, \tag{1.12}$$

$$x(0,s) = x_0(s) \in W_2',$$

are bounded, and the zero solutions are stable. Moreover, if $B(t, \partial x/\partial t) \equiv \partial x/\partial t$ and $\Phi'_x(t, x)$ satisfies a Hölder condition, then problem (1.12) is nonlocally solvable, and if (1.5) is satisfied, then the zero solution of problem (1.12) is also asymptotically stable.

1.4. Let us give another example. Let $H = E^n$ and put

$$A(t) \equiv (a_{ij}(t))_{i,j=1,\dots,n}, \qquad B(t, \dot{x}) \equiv \dot{x} + (g_i(t, \dot{x}))_{i=1,\dots,n},$$

where $a_{ij} = a_{ji}$, $\sum a_{ij}\xi_i\xi_j \geq 0$, $\sum g_i(t, \dot{x})\dot{x} \geq 0$. Suppose that the $a_{ij}(t)$ are differentiable, that these derivatives are bounded by the function $\alpha(t)$, and that (1.4) is satisfied. Then A and B will satisfy, respectively, conditions (A) and (B_0). Let $\psi(t, x_1, \dots, x_n)$ be a positive (or else $\lim_{\|x\|\to\infty} \psi(t, x) = \infty$) function differentiable with respect to all arguments. In addition, let $\psi'_x(t, x) \leq \alpha(t)\psi(t, x)$. Putting $F(t, x) = \psi(t, x_1, \dots, x_n)$, from Theorem 1 we obtain that the solution of the problem

$$\dot{x}_i + g_i(t, x_1, \dots, \dot{x}_n) + \sum_{i=1}^{n} a_{ij}(t)x_j + \frac{\partial\psi}{\partial x_i}(t, x_1, \dots, x_n) = 0,$$

$$x_i(0) = x_i^{(0)}$$

is bounded for $t \in [0, \infty)$, and its zero solution is stable.

2. **Stability of solutions.** Let $x(t), y(t) \in S_r$ ($\|A^{1/2}x\| \leq r$) be solutions of the problems

$$B(t, \dot{x}) + A(t)x = f(t, x), \qquad x(0) = x_0, \tag{2.1}$$

$$B(t, \dot{y}) + A(t)y = f(t, y), \qquad y(0) = y_0. \tag{2.2}$$

Here the nonlinear operator $f(t, x)$ $(0 \leq t < \infty,\ x \in S_r)$ is not necessarily potential.

Theorem 3. Let the operator $A(t)$ satisfy condition (A). Let the operator $B(t, \dot{x})$ $(0 \leq t < \infty,\ \dot{x} \in D(B))$ satisfy the condition

$$(B(t, \dot{x}) - B(t, \dot{y}), \dot{x} - \dot{y}) \geq \frac{1}{2}\|\dot{x} - \dot{y}\|^2.$$

Suppose that for $t \in [0, \infty)$, $x, y \in S_r$, one has

$$\|f(t, x) - f(t, y)\| \leq K(t, r)\|A^{1/2}(t)(x - y)\|, \tag{2.3}$$

where

$$\int_0^\infty K^2(t,r)\,dt < \infty.$$

Finally, let $x_0 - y_0 \in D(A^{1/2}(0))$. Then the inequality holds

$$\|A^{1/2}(t)[x(t)-y(t)]\| \le c\|A^{1/2}(0)(x_0-y_0)\| \exp\left[2\int_0^t \alpha(s)\,ds\right] \quad (t \ge 0). \quad (2.4)$$

2.1. It follows from this theorem that, if (1.4) is satisfied, then the solution of problem (2.1) is unique and stable, i.e. sufficient smallness of $\|A^{1/2}(0)(x_0-y_0)\|$ implies sufficient smallness of $\|A^{1/2}(t)(x-y)\|$; and if (1.5) is satisfied, then this solution is asymptotically stable, i.e.

$$\lim_{t\to\infty} \|A^{1/2}(t)(x-y)\| = 0.$$

The stability conditions obtained from Theorem 3 are close to the known ones obtained by a number of authors, beginning with the fundamental work of M. G. Krein ([4]).

2.2. We have given an example of the operator $A(t)$. An example of the operator $B(t,\dot{x})$ may be the operator (1.9) under the conditions

$$\sum b_{ij}\xi_i\xi_j \ge \delta_1 \sum \xi_i^2, \qquad g(t,\dot{x})\dot{x} \ge \delta_2 \dot{x}^2 \quad (\delta_1+\delta_2 = 1/2). \qquad (2.5)$$

Now let us give examples of the operator $f(t,x)$. Let $H = L^2(G)$ and $f(t,x) = \varphi(t,x)$, with $|\varphi'_x(t,x)| \le M_1|x|^m$. Then it is easy to verify that the scalar function $\varphi(t,x)$ satisfies condition (2.3). Thus, from Theorem 3 it follows that the solution of the problem

$$B[t, \partial x(t,s)/\partial t] + A(t)x(t,s) = \varphi[t, x(t,s)], \qquad (2.6)$$

$$x(0,s) = x_0(s)$$

is stable if (1.4) is fulfilled, and asymptotically stable under condition (1.5). Here the operators $A(t)$ and $B(t, \partial x/\partial t)$ are defined by formulas (1.8), (1.9).

3. **Stabilization of solutions.** Let $y^*(t)$ be a differentiable solution from $D(B)$ of the equation

$$A(t)y + P(t,y) = 0. \tag{3.1}$$

Then $y^*(t)$ is simultaneously a solution of the equation

$$B(t,\dot{y}) + A(t)y + P(t,y) = B(t,\dot{y}). \tag{3.2}$$

We shall say that (cf. ([5],[6])) the solutions of equation (1.1) are stabilized to the solution of equation (3.1) if

$$\lim_{t\to\infty} \left\| A^{1/2}(t)\,[x(t) - y^*(t)] \right\| = 0.$$

Theorem 4. Let the operator $A(t)$ satisfy condition (A). Let the operator $B(t,\dot{x})$ $(0 \le t < \infty,\ \dot{x} \in D(B))$ satisfy the condition

$$\left(B(t,\dot{x}) - B(t,\dot{y}), \dot{x} - \dot{y}\right) \ge \|\dot{x} - \dot{y}\|^2.$$

Let the operator $P(t,x)$ satisfy condition (2.3). Then

$$\left\| A^{1/2}(t)\,[x(t) - y^*(t)] \right\|^2 \le \left\| A^{1/2}(0)(x_0 - y_0^*) \right\| \exp\left\{ \int_0^t [\alpha(s) + k^2(s)]\,ds \right\} + \int_0^t \varepsilon(s)\, \exp\left\{ \int_0^t [\alpha(\tau) + k^2(\tau)]\,d\tau \right\} ds = \varepsilon_0(t),$$

where

$$\varepsilon(s) \equiv \left\| \dot{B}[s, \dot{y}^*(s)] \right\|.$$

From this theorem it is seen that conditions under which $\varepsilon_0(t) \to \infty$ as $t \to \infty$ are conditions for the stabilization of all solutions of equation (1.1) to the solution $y^*(t)$ of equation (3.1). The specification of this assertion for the case of equation (1.12) is carried out according to the scheme of M. I. Vishik and L. A. Lyusternik ([5]), and the requirements on the function $\gamma(t)$ formulated in those theorems should be replaced by the same requirements on the function $\gamma_1(t) = -\alpha(t) - k^2(t)$. The nonlinear term $\Phi(t,x)$ additionally satisfies the condition

$$|\Phi''_{xx}(t,x)| \le M_2 x^{m-1}.$$

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Note: Figure translations are in progress. See original paper for figures.