EXACT SOLUTION OF A PROBLEM OF PLASTIC FLOW IN A THIN LAYER OVER ELASTIC SURFACES

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Abstract

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EXACT SOLUTION OF A PROBLEM OF PLASTIC FLOW IN A THIN LAYER OVER ELASTIC SURFACES

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Let us consider a layer of plastic material which occupies in the xy-plane a certain region S and is enclosed between the surfaces

$$z_1 = f_1(x, y) - \int_0^t V_0(t)\, dt \quad \text{and} \quad z_2 = f_2(x, y),$$

bounding two elastic bodies; these bodies, approaching one another with velocity $V_0(t)$, cause the layer to spread. We shall assume the functions $f_1(x, y)$ and $f_2(x, y)$ to be smooth, and their first derivatives with respect to each of the coordinates to be of order $h_0/L \ll 1$, where $h_0 = z_1 - z_2$, and L is the characteristic size of the region S.

The boundary Γ of the region may be formed by places of abrupt change in the thickness of the layer or by grooves in the bodies, into which the material of the layer may flow, forming ribs; we shall regard Γ as a piecewise-smooth curve $y_0 = \varphi(x_0)$.

Under these conditions the pressure in the layer is determined from the equation (1)

$$(\operatorname{grad} P)^2 = \frac{4\tau_s^2}{h^2} \tag{1}$$

and the boundary condition

$$\text{on } \Gamma \qquad P = 2\sigma_s, \tag{2}$$

where P is the pressure; $\sigma_s = \sqrt{3}\tau_s$ is the yield strength of the material of the layer; h is its thickness, which in the problem under consideration may be written in the form

$$h = h_0 + w_1 + w_2;$$

in this expression w_1 and w_2 denote the normal displacements of the boundary points of the elastic bodies; they are taken as positive if directed into the bodies.

Assuming that the corresponding Green' s functions exist, we obtain

$$w_1 = \iint_S K_1(x, y, \xi, \eta) P(\xi, \eta)\, d\xi\, d\eta,$$

$$w_2 = \iint_S K_2(x, y, \xi, \eta) P(\xi, \eta)\, d\xi\, d\eta;$$

adding these equalities term by term and denoting $w = w_1 + w_2$, $H(x, y, \xi, \eta) = 2\tau_s[K_1(x, y, \xi, \eta) + K_2(x, y, \xi, \eta)]$, $u = P/2\tau_s$, we find

$$w = \iint_S H(x, y, \xi, \eta) u(\xi, \eta)\, d\xi\, d\eta; \tag{3}$$

equation (1) and boundary condition (2) take the form

$$(\operatorname{grad} u)^2 = \frac{1}{(h_0 + w)^2}; \tag{4}$$

$$\text{on } \Gamma \qquad u = \sqrt{3}. \tag{5}$$

The problem is reduced to the joint solution of the system (3), (4) under condition (5); in [2] it was proposed to solve this system by the method of successive approximations. We put $w_0 = 0$ and from (4) find u_0; substituting this into (3), we find w_1, in accordance with which from (4) we determine u_1, etc. Below an example is considered for which an exact solution of the system (3)–(5) is constructed and it is shown how rapidly the approximation process converges.

Put $h_0 = h_1 = \text{const}$ and $H(x, y, \xi, \eta) = \lambda\delta(\xi - x, \eta - y)$; substituting this into (3), we find

$$w(x, y) = \lambda \iint\limits_S \delta(\xi - x, \eta - y) u(\xi, \eta)\, d\xi\, d\eta = \lambda u(x, y);$$

hence it is seen that the adopted Green' s function describes the Winkler model of an elastic foundation, and from (4) we then obtain

$$(\operatorname{grad} u)^2 = 1/(h_1 + \lambda u)^2, \tag{6}$$

and thus the problem is reduced to the Cauchy problem for equation (6) under condition (5). The corresponding characteristic system in Monge' s notation has the form

$$\frac{dx}{ds} = 2p, \quad \frac{dy}{ds} = 2q, \quad \frac{du}{ds} = 2(p^2 + q^2), \quad \frac{dp}{ds} = -\frac{2\lambda p}{(h_1 + \lambda u)^3},$$

$$\frac{dq}{ds} = -\frac{2\lambda q}{(h_1 + \lambda u)^3}$$

and the solution

$$(h_1 + \lambda u)^2 = (h_1 + \lambda\sqrt{3})^2 \pm (x_0 - x)\frac{2\lambda\sqrt{1 + y_0'^2}}{y_0'}, \tag{7}$$

$$(x_0 - x) = -(y_0 - y)y_0', \qquad y = \varphi(x_0);$$

hence we find the final expression for the displacements

$$\bar{w} = \delta u = \left[(1 + \delta\sqrt{3})^2 \pm \frac{x_0 - x}{h_1}\frac{2\delta\sqrt{1 + y_0'^2}}{y_0'}\right]^{1/2} - 1, \tag{8}$$

where $\bar{w} = w/h_1$, $\delta = \lambda/h_1$. In what follows we shall assume $\bar{w}_{\max} < 1$; since $u_{\max} \approx L/h_1$, the stated requirement will be satisfied if $\delta L/h_1 < 1$.

We now use the method of approximations to solve the problem. Setting $w_0 = 0$ and substituting this into (4), we obtain

$$\bar{w}_1 = \delta u_1 = \delta\sqrt{3} \pm \delta\frac{x_0 - x}{h_1}\frac{\sqrt{1 + y_0'^2}}{y_0'}, \tag{9}$$

which coincides with the principal term in the formal expansion of (8) in the parameter δ. We construct the following approximations for a circular domain; from (7) for this case we obtain

$$\bar{w} = \delta u = \left[(1 + \delta\sqrt{3})^2 + 2\delta t\right]^{1/2} - 1, \tag{10}$$

where $t = (R - r)/h_1$, R is the radius of the circle, and from (9)

$$\bar{w}_1 = \delta u_1 = \delta(\sqrt{3} + t).$$

Substituting this into (4), we find the second approximation

$$\bar{w}_2 = \delta u_2 = \delta\sqrt{3} + \ln\left(1 + \frac{\delta t}{1 + \delta\sqrt{3}}\right);$$

analogously, the third approximation is obtained

$$\bar{w}_3 = \delta u_3 = \delta\sqrt{3} + \int_1^{1+\beta t} \frac{dx}{1 + \frac{1}{\alpha}\ln x} =$$

$$= \delta\sqrt{3} + \sum_{m=0}^{\infty}(-1)^m \left[\frac{x}{m+1}\sum_{k=0}^{m}(-1)^k(m+1)m(m-1)\dots\right.$$

$$\left.\dots(m-k+1)\left(\frac{1}{\alpha}\ln x\right)^{m-k}\right]_{x=1}^{x=1+\beta t},$$

where $\alpha = 1 + \delta\sqrt{3}$, $\beta = \delta/(1+\delta\sqrt{3})$, etc. Let, for example, $\delta = 1/30$, $\delta R/h_1 = 0.7$; then for the center of the circle we successively find $\bar{w}_1 = 0.758$, $\bar{w}_2 = 0.556$, $\bar{w}_3 = 0.614$, whereas the exact solution gives the result $\bar{w} = 0.59$. The first approximation is overestimated in comparison with the exact one by 28%, the second is underestimated by 6%, and the third exceeds the exact one by 4%. Hence it is clear that, in estimating the displacements w (in other words, the deviations from the prescribed value of the layer thickness due to elastic deformations of the tool), it is quite permissible to use the first approximation, while the second gives a practically exact result. The parameters have been chosen so as to bring the calculation closer to the real conditions of pressure-working technology [3].

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CITED LITERATURE

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