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1965

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Abstract

Full Text

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FINITE GROUPS WITH A PAIR OF NONCONJUGATE NILPOTENT MAXIMAL SUBGROUPS

(Presented by Academician A. I. Mal' tsev, 19 XI 1964)

This article gives a description of finite nonnilpotent groups having two conjugacy classes of nilpotent maximal subgroups; moreover, it is established that finite groups having more than two conjugacy classes of nilpotent maximal subgroups are nilpotent. This main result of the article is then applied to the description of certain other classes of finite groups. In this connection the role in the general theory of finite groups is clarified for the class of so-called groups of type A, singled out by A. A. Kulakov and S. A. Chunikhin ([1]) in the study of the question of the existence in groups of subgroups of composite order.

Definition ([1])**.** Groups of type A are nonnilpotent groups of order pq^β (p and q are primes), in which the Sylow subgroup of order q^β is an invariant elementary abelian group and β is the exponent of the number q modulo p.

It is not difficult to show that the class of groups of type A is the intersection of the classes of Schmidt groups and Frobenius groups.

Theorem 1. *Let $\mathfrak{G}$ be a finite nonnilpotent group. Then:*

1) $\mathfrak{G}$ cannot have more than two conjugacy classes of nilpotent maximal subgroups;

2) $\mathfrak{G}$ has two conjugacy classes of nilpotent maximal subgroups if and only if it is an extension of a nilpotent group $\mathfrak{N}$ by a group $\mathfrak{A}$ of type A, and in this case every element of $\mathfrak{A}$ acts identically on the elements of $\mathfrak{N}$ whose orders are relatively prime to it.

Thus, if $\mathfrak{H}$ and $\mathfrak{F}$ are a pair of nonconjugate nilpotent maximal subgroups of a finite nonnilpotent group $\mathfrak{G}$, then:

1) one of the subgroups $\mathfrak{H}$ and $\mathfrak{F}$, and only one, is invariant in $\mathfrak{G}$;

2) $\mathfrak{H} \cap \mathfrak{F}$ is invariant in $\mathfrak{G}$;

3) $\mathfrak{G}/\mathfrak{H} \cap \mathfrak{F}$ is a group of type A;

4) there exists a system $\mathfrak{S}$ of representatives of adjacent classes of $\mathfrak{G}$ modulo $\mathfrak{H} \cap \mathfrak{F}$ such that every element of $\mathfrak{S}$ is permutable with each element of order relatively prime to it from $\mathfrak{H} \cap \mathfrak{F}$;

5) $\mathfrak{H} \cap \mathfrak{F}$ is a single normal divisor of the group $\mathfrak{G}$, satisfying conditions 3) and 4);

6) $\mathfrak{G} = \mathfrak{L} \times \mathfrak{B}$, where $\mathfrak{L}$ is a nilpotent group (even $\mathfrak{L} \subseteq \mathfrak{H} \cap \mathfrak{F}$), and $\mathfrak{B}$ is a group which is a semidirect product of two of its Sylow subgroups and has a pair of nonconjugate nilpotent maximal subgroups;

7) the group $\mathfrak{G}$ is covered by its nilpotent maximal subgroups (i.e., it is the set-theoretic union of the subgroups conjugate to $\mathfrak{H}$ and $\mathfrak{F}$).

Let us note that for an arbitrary finite nilpotent group $\mathfrak{N}$ and an arbitrary group $\mathfrak{A}$ of type A there always exists an extension of $\mathfrak{N}$ by

with the aid of $\mathfrak{N}$, described in Theorem 1; such will be, for example, the direct product of $\mathfrak{N}$ and $\mathfrak{A}$. Moreover, the direct product of a nilpotent group and a Schmidt group is also a group with two classes of conjugate nilpotent maximal subgroups (the groups are assumed finite). The converse assertion, however, will be false, as is directly seen from the example of the group given by the defining relations

$$A^4 = 1, \quad B^2 = A^2, \quad BA = A^{-1}B, \quad C^3 = 1, \quad AC = CA, \quad BC = C^{-1}B.$$

This example also shows that the extension described in Theorem 1 is not always split.

In the proof of Theorem 1, Theorem 2 of the paper ([2]) of J. Thompson was used, as well as Theorem 9 of the author' s paper ([3]). This latter theorem admits the following substantial refinement.

Theorem 2. *For a finite group $\mathfrak{G}$ the following conditions are equivalent:*

1) *$\mathfrak{G}$ has at least two classes of conjugate Sylow maximal subgroups;*

2) *all Sylow subgroups of $\mathfrak{G}$ are maximal;*

3) *all maximal subgroups of $\mathfrak{G}$ are Sylow;*

4) *$\mathfrak{G}$ is nonsimple and has a pair of maximal subgroups of relatively prime orders;*

5) *$\mathfrak{G}$ has a Sylow maximal subgroup of prime order;*

6) *$\mathfrak{G}$ is a cyclic group of order pq (p and q are distinct primes) or a group of type A.*

The equivalence of conditions 3) and 6) was established earlier by A. A. Kulakov and S. A. Chunikhin ([1]).

Imposing on a finite group conditions of mutual coprimeness of some of its maximal subgroups, we obtain various classes of groups very close to the classes of groups described in Theorem 2.

Theorem 3. *The following conditions on a finite group $\mathfrak{G}$ are equivalent:*

1) *$\mathfrak{G}$ has a nilpotent maximal subgroup mutually coprime with every maximal subgroup of the group $\mathfrak{G}$ not conjugate to it;*

2) $\mathfrak{G}$ *has a maximal subgroup mutually coprime with every other maximal subgroup of the group* $\mathfrak{G}$*;*

3) *any two maximal subgroups of the group* $\mathfrak{G}$ *are mutually coprime;*

4) $\mathfrak{G}$ *is nonsimple and has a maximal subgroup mutually coprime with every maximal subgroup of the group* $\mathfrak{G}$ *not conjugate to it;*

5) $\mathfrak{G}$ *is a cyclic group of order* p^α*, or a group of order* pq *(*p, q *are distinct or equal primes), or a group of type* A*.*

The equivalence of conditions 1) and 5) of this theorem generalizes a known theorem of S. A. Chunikhin ([4], Satz III).

Let us note in conclusion that for any pair of distinct primes p and q there are two and only two groups of type A whose order is divisible by pq: one of order pq^β, where β is the exponent of the number q modulo p, the other of order $p^\alpha q$, where α is the exponent of the number p modulo q.

This follows directly from Theorem 3 of the paper ([5]) of Yu. A. Gol' fand.

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Received
10 XI 1964

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Note: Figure translations are in progress. See original paper for figures.