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Abstract

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MATHEMATICS

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ON ONE PROPERTY OF LEVEL LINES UNDER UNIVALENT CONFORMAL MAPPINGS

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Let S be the class of regular and univalent functions

$$f(x) = z + a_2 z^2 + a_3 z^3 + \cdots$$

and let $L(f, r)$ be the image of the circle $|z| = r$ under its mapping in the disk $|z| < 1$ by the function $f(z)$ (a level line). In papers ([1–3]) the connection was investigated between the property of the level line $L(f, r)$ of being star-shaped or convex and the value of the modulus of the function $f(z)$. In the present work a connection is established between the star-shapedness of an arc of a level line and the value of the modulus of the derivative $f'(z)$. For this purpose, on the basis of the method set forth in paper ([2]), the domain D_r of values of the functional

$$I(f) = \ln |f'(z)| - i \arg(z f''(z)/f'(z)), \qquad |z| = r,$$

in the class S, is found.

The domain D_r is closed, bounded, convex, and symmetric with respect to the real axis. In the upper half-plane it is bounded by a continuous curve, whose endpoints lie on the real axis and which consists of arcs of four analytic curves. For $r > (\sqrt{3} - 1)/\sqrt{2}$, among these arcs there is a line segment; for $r \leq (\sqrt{3} - 1)/\sqrt{2}$ the segment contracts to a point. The equations of all these arcs have been found in explicit analytic form.

The investigation carried out of the domain of values of the functional $I(f)$ makes it possible to formulate the following theorem:

Theorem. *For each r there exist numbers $\alpha_1(r) < 1$, $\beta_1(r) > 1$ such that the arc of the level line $L(f, r)$ is star-shaped for any function $f(z) \in S$ if and only if*

$$\text{either} \quad \alpha_1(r)\underline{R_1(r)} < |f'(z)| < \overline{R_1(r)},$$

$$\text{or} \quad \underline{R_1(r)} < |f'(z)| < \overline{R_1(r)}\beta_1(r),$$

where $\alpha_1(r)$, $\beta_1(r)$ are determined from certain equations;

$$\underline{R_1(r)} = \frac{1-r}{(1+r)^3}, \qquad \overline{R_1(r)} = \frac{1+r}{(1-r)^3}$$

are, respectively, the exact lower and upper bounds for $|f'(z)|$ in the class S.

From this theorem it follows that for any $r < 1$

$$\alpha_1(r) \leq \alpha_1 = \frac{9}{16} e^{4\sqrt{2}\,\operatorname{arctg}(1-2\sqrt{2})/(1+2\sqrt{2})} = 0.047\ldots,$$

where

$$\lim_{r\to 1} \alpha_1(r) = \alpha_1.$$

Thus, for any function $f(z) \in S$, an arc of the level line $L(f, r)$ for which the inequalities

$$\alpha_1\underline{R_1(r)} < |f'(z)| < \overline{R_1(r)}$$

hold will be star-shaped for every r, $0 < r < 1$.

Correspondingly, $\beta_1(r) \geq 1$, and

$$\lim \beta_1(r) = 1.$$

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Note: Figure translations are in progress. See original paper for figures.