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Abstract

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ON ONE VARIANT OF CONSTRUCTIVE PREDICATE CALCULUS WITHOUT STRUCTURAL RULES OF INFERENCE

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Let us list the principal syntactic signs and expressions used below, with an explanation of their meaning: $\mathfrak{A}, \mathfrak{B}$ are arbitrary logical formulas (abbreviated: formulas); $\mathfrak{C}$ is either an elementary formula, or a formula beginning with the sign $\neg$, or a formula $\forall x \neg \mathfrak{A}$; $\mathfrak{D}$ is an arbitrary non-elementary formula not beginning with the sign $\neg$ and not being a formula $\forall x \neg \mathfrak{A}$; $\Gamma, \Gamma', \Gamma''$ are arbitrary formula chains; Θ is an arbitrary formula or the empty word; $x, y, z, x_1, y_1, z_1, \ldots$ are arbitrary object variables. We shall also use certain terms and notations from the paper ([1])*, as well as the terminology of the book ([2]).

Let J_0 denote the calculus specified by the axiom scheme

$$\Gamma' \mathfrak{A} \Gamma'' \to \mathfrak{A}$$

and by the following rules of inference:

1. $\dfrac{\Gamma \mathfrak{A} \to \mathfrak{B}}{\Gamma \to (\mathfrak{A} \supset \mathfrak{B})}$ 2a. $\dfrac{\Gamma' \Gamma'' \to \mathfrak{C};\ \Gamma' \mathfrak{B} \Gamma'' \to \Theta}{\Gamma' (\mathfrak{C} \supset \mathfrak{B}) \Gamma'' \to \Theta}$

3. $\dfrac{\Gamma \to \mathfrak{A};\ \Gamma \to \mathfrak{B}}{\Gamma \to (\mathfrak{A} \& \mathfrak{B})}$ 2b. $\dfrac{\Gamma' (\mathfrak{D} \supset \mathfrak{B}) \Gamma'' \to \mathfrak{D};\ \Gamma' \mathfrak{B} \Gamma'' \to \Theta}{\Gamma' (\mathfrak{D} \supset \mathfrak{B}) \Gamma'' \to \Theta}$

5a. $\dfrac{\Gamma \to \mathfrak{A}}{\Gamma \to (\mathfrak{A} \vee \mathfrak{B})}$ 4. $\dfrac{\Gamma' \mathfrak{A} \mathfrak{B} \Gamma'' \to \Theta}{\Gamma' (\mathfrak{A} \& \mathfrak{B}) \Gamma'' \to \Theta}$

5b. $\dfrac{\Gamma \to \mathfrak{B}}{\Gamma \to (\mathfrak{A} \vee \mathfrak{B})}$ 6. $\dfrac{\Gamma' \mathfrak{A} \Gamma'' \to \Theta;\ \Gamma' \mathfrak{B} \Gamma'' \to \Theta}{\Gamma' (\mathfrak{A} \vee \mathfrak{B}) \Gamma'' \to \Theta}$

7. $\dfrac{\Gamma \mathfrak{A} \to}{\Gamma \to \neg \mathfrak{A}}$ 8a. $\dfrac{\Gamma' \Gamma'' \to \mathfrak{C}}{\Gamma' \neg \mathfrak{C} \Gamma'' \to \Theta}$

9. $\Gamma \to \mathfrak{A}$ 8b. $\dfrac{\Gamma' \neg \mathfrak{D} \Gamma'' \to \mathfrak{D}}{\Gamma' \neg \mathfrak{D} \Gamma'' \to \Theta}$

x and y satisfy

$$\frac{\text{condition } (*)}{\Gamma \to \forall x\, [\mathfrak{A}]^y_x}$$

11. $\Gamma \to [\mathfrak{A}]^x_z$

$$\frac{x \text{ and } z \text{ satisfy condition } (**)}{\Gamma \to \exists x \mathfrak{A}}$$

10. $\Gamma' [\mathfrak{A}]^x_z\, \forall x \mathfrak{A}\, \Gamma'' \to \Theta$

$$\frac{x \text{ and } z \text{ satisfy condition } (**)}{\Gamma' \forall x \mathfrak{A} \Gamma'' \to \Theta}$$

12. $\Gamma' \mathfrak{A} \Gamma'' \to \Theta$

$$\frac{x \text{ and } y \text{ satisfy condition } (*)}{\Gamma' \exists x\, [\mathfrak{A}]^y_x \Gamma'' \to \Theta}$$

* As V. A. Matulis has informed me, in the paper ([1]), in the formulation of the definition of a pure sequent the following condition was omitted: in the sequent S there cannot be a quantifier complex in whose scope there is another quantifier complex with the same proper variable as that of the first quantifier complex.

The conditions (*) and (**) are exactly the same as in ([1]).

Theorem 1. *The calculus J_0 and the constructive calculus $G1$ (see ([2]), §77) are not equipollent. Moreover, if S is a pure sequent, then it is derivable in J_0 if and only if it is derivable in the constructive calculus $G1$.*

In constructive logic, the passage from the constructive calculus $G1$ and the equipollent constructive calculi $G2$ and $G3$ (see ([2]), §§78 and 80) to the calculus J_0 pursues the same aim as the passage in classical logic from the classical calculi $G1$, $G2$, and $G3$ to the calculus E_0, constructed in ([1]): the calculus J_0 is better adapted to the search for a derivation in the constructive predicate calculus than the constructive calculus $G3$.

The principal differences between the calculus J_0 and the constructive calculus $G3$ are as follows: 1) in the calculus J_0 there are no structural rules of inference and the notion of an initial sequent is not used; 2) the principal formulas of the inference rules of the calculus J_0 are repeated in the premisses only of those rules for which such duplication of the principal formula is indeed necessary; 3) the restrictions on variables in the quantifier rules of the calculus J_0 are stronger than in the constructive calculus $G3$.

The necessity of the above-mentioned duplication of the principal formula in the rules 2 , 8 , and 10 is seen as follows. Consider the rules for transforming sequents:

$$2^*. \quad \frac{\Gamma'\Gamma'' \to \mathfrak{D};\ \Gamma'\mathfrak{B}\Gamma'' \to \Theta}{\Gamma'(\mathfrak{D} \supset \mathfrak{B})\Gamma'' \to \Theta}$$

$$8^*. \quad \frac{\Gamma'\Gamma'' \to \mathfrak{D}}{\Gamma'\neg\mathfrak{D}\Gamma'' \to \Theta}$$

$$10^*. \quad \frac{\Gamma'[\mathfrak{A}]^x_z\Gamma'' \to \Theta}{x \text{ and } z \text{ satisfy}}$$

$$\frac{\text{the condition } (**)}{\Gamma'\forall x\,\mathfrak{A}\Gamma'' \to \Theta}$$

Let J_0^* (respectively J_0^{**}, respectively J_0^{***}) denote the calculus obtained from J_0 by replacing rule 2 by rule 2^* (respectively, rule 8 by rule 8^*, rule 10 by rule 10^*). The calculi J_0^*, J_0^{**}, and J_0^{***} are not equipollent with the calculus J_0. Indeed, in J_0^* the sequent

$$(((\mathfrak{A} \supset \mathfrak{B}) \vee \mathfrak{A}) \supset \mathfrak{B}) \to \mathfrak{B},$$

where $\mathfrak{A}$ and $\mathfrak{B}$ are elementary formulas, is not derivable; in J_0^{**} the sequent

$$\to \neg\neg(\mathfrak{A} \vee \neg\mathfrak{A}),$$

where $\mathfrak{A}$ is an elementary formula, is not derivable; in J_0^{***} the sequent

$$\forall x((\mathfrak{A} \supset [\mathfrak{A}]^x_{z_1}) \supset \mathfrak{A}) \to [\mathfrak{A}]^x_{z_2},$$

where $\mathfrak{A}$ is an elementary formula containing a free occurrence of the variable x, and z_1 and z_2 are variables distinct from one another and from x, is not derivable. At the same time, the three indicated sequents are derivable in J_0, and the last sequent is derivable even in J_0^*. We note that if to J_0^*, J_0^{**}, and J_0^{***} one adjoins the structural rule of contraction of repetitions, then the calculi thereby obtained turn out to be equipollent with the calculus J_0.

All inference rules of the calculus J_0, except rules 2a, 2 , 5a, 5 , 8a, 8 , and 11, are invertible.*

The realizations of rules 2a, 2 , 8a, and 8 turn out to be invertible in those cases where Θ is the empty word. In connection with rules 2a and 2 , we note that if the conclusion of some realization of rule 2a (rule 2) is derivable, then the right premiss of this realization of rule 2a (respectively, of rule 2) is also derivable.

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* An inference rule is called invertible if, in each of its realizations (i.e. in each figure obtained from the inference rule as the result of substituting concrete formulas, concrete formula strings, and concrete variables for the syntactic signs and expressions symbolizing, respectively, formulas, formula strings, and variables), a sequent standing below the line is derivable if and only if all sequents standing above the line are derivable.

2. For the proof of Theorem 1 an auxiliary calculus J_0^+ is introduced, which is obtained from the calculus J_0 by replacing rules $2a, 2b, 8a$, and $8b$, respectively, by the rules

$$2a^+.\quad \frac{\Gamma'\Gamma'' \to \mathfrak{C}_1;\ \Gamma'\mathfrak{B}\Gamma'' \to \Theta}{\Gamma'(\mathfrak{C}_1 \supset \mathfrak{B})\Gamma'' \to \Theta} \qquad 8a^+.\quad \frac{\Gamma'\Gamma'' \to \mathfrak{C}_1}{\Gamma'\neg\mathfrak{C}_1\Gamma'' \to \Theta}$$

$$2b^+.\quad \frac{\Gamma'(\mathfrak{D}_1 \supset \mathfrak{B})\Gamma'' \to \mathfrak{D}_1;\ \Gamma'\mathfrak{B}\Gamma'' \to \Theta}{\Gamma'(\mathfrak{D}_1 \supset \mathfrak{B})\Gamma'' \to \Theta} \qquad 8b^+.\quad \frac{\Gamma'\neg\mathfrak{D}_1\Gamma'' \to \mathfrak{D}_1}{\Gamma'\neg\mathfrak{D}_1\Gamma'' \to \Theta}$$

Here $\mathfrak{C}_1$ is an arbitrary formula beginning with the sign $\neg$, or a formula $\forall x\neg\mathfrak{A}$, while $\mathfrak{D}_1$ is an arbitrary formula not beginning with the sign $\neg$ and not being a formula $\forall x\neg\mathfrak{A}$.

Theorem 2. a) *The calculus J_0^+ and the constructive calculus $G1$ are equivalent in scope. Moreover, if S is a pure sequent, then it is derivable in J_0 if and only if it is derivable in the constructive calculus $G1$.* b) *The calculus J_0^+ and the calculus J_0 are equivalent in scope.*

The proof of part a) is carried out as follows:

1) A simple method is indicated for transforming any derivation tree in the calculus J_0^+ into a derivation tree in the constructive calculus $G1$ having the same final sequent as the first derivation tree.

2) An auxiliary calculus is introduced, obtained from J_0^+ by adjoining the structural rules $\to Y_1 Y \to$, $\Pi \to$ (see (2), §77) and Gentzen' s structural rule of contraction of repetitions (see (3)):

$$\frac{\Gamma\mathfrak{A}\Gamma'\mathfrak{A}\Gamma'' \to \Theta}{\Gamma\mathfrak{A}\Gamma'\Gamma'' \to \Theta} \qquad C' \to .$$

It is proved that, whatever pure sequent S may be, any derivation of it in the constructive calculus $G1$ can be transformed into a derivation of the sequent S in the calculus thus obtained.

3) It is proved that every pure* derivation in the indicated extension of the calculus J_0^+ can be transformed into a pure derivation in the calculus J_0^+ with the same final sequent as the first derivation. In proving item 3), the greatest difficulty is the proof of the eliminability of applications of the structural rule $C' \to$. In this proof the following concepts and considerations are used.

We shall say that a formula chain Δ_1 is similar to a formula chain Δ_2 if exactly the same formulas occur in Δ_1 as in Δ_2.

The relation "the formula chain Δ majorizes the formula $\mathfrak{A}$" (symbolic notation $\Delta \succ \mathfrak{A}$) is defined by the following generating rules: $\mathfrak{A} \succ \mathfrak{A}$; if $\Delta \succ \mathfrak{M}$, then $\Delta \succ \neg\neg\mathfrak{M}$; if $\Delta \succ \mathfrak{M}$, then $\Delta \succ \neg\forall x[\neg\mathfrak{M}]^y_x$; if $\Delta \succ \mathfrak{M}$, then $\Delta \succ (\mathfrak{M} \supset \mathfrak{N})$; if $\Delta \succ \neg\mathfrak{M}$, then $\Delta \succ (\mathfrak{M} \supset \mathfrak{N})$; if $\Delta_1 \succ \mathfrak{M}$, $\Delta_2 \succ \mathfrak{N}$, and Δ is a formula chain similar to $\Delta_1\Delta_2$, then $\Delta \succ (\mathfrak{M}\&\mathfrak{N})$; if $\Delta \succ \mathfrak{M}$, then $\Delta \succ (\mathfrak{M} \vee \mathfrak{N})$; if $\Delta \succ \mathfrak{N}$, then $\Delta \succ (\mathfrak{M} \vee \mathfrak{N})$; if $\Delta \succ \mathfrak{M}$, then $\Delta \succ \exists x[\mathfrak{M}]^y_x$, where x and y are variables such that y is free for x in $\mathfrak{M}$.

It is easy to see that the majorization relation is algorithmically decidable.

Lemma 1. a) *If $\Delta \succ \mathfrak{A}$ and $\Delta'\mathfrak{A}\Delta'' \succ \mathfrak{B}$, then $\Delta\Delta'\Delta'' \succ \mathfrak{B}$;* b) *if $\Delta \succ \mathfrak{A}$, then the sequent $\Delta \to \mathfrak{A}$ is derivable in the calculus J_0^+.*

Lemma 2. *If one can construct a pure derivation tree in the calculus J_0^+ for the sequent $\Gamma'\mathfrak{A}\Gamma'' \to \Theta$ (a pure derivation tree in the calculus J_0^+ for the sequent $\Gamma'\mathfrak{A}\Gamma'' \to \Theta$ without such applications of the rules $2b^+$ and $8b^+$ in which $\mathfrak{D}_1$ is an elementary formula) and Δ is a formula chain such that: 1) every member of this formula chain is a member of the formula chain $\Gamma'\Gamma''$, and 2) $\Delta \succ \mathfrak{A}$, then one can construct a pure derivation tree in the calculus J_0^+ of the seq—*

* A derivation tree is called pure if it has the variable-purity property (see (2), §78).

of the sequent $\Gamma'\Gamma'' \to \Theta$ (respectively, a pure derivation tree in the calculus J_0^+ of the sequent $\Gamma'\Gamma'' \to \Theta$ without such applications of the rules $2\delta^+$ and $8\delta^+$, in which $\mathfrak{D}_1$ is an elementary formula).

Item b) of Theorem 2 is easily proved by means of a method analogous to that used in paper ([4]), and by means of Lemma 2.

3. In the actual search for a derivation in the constructive predicate calculus, it is expedient (in order to shorten the search process) to adjoin to the calculus J_0 certain admissible rules of inference, for example the rule

$$\frac{\Gamma''\mathfrak{A}_1\mathfrak{A}_2 \dots \mathfrak{A}_{n-1}(\mathfrak{A}_n \supset \mathfrak{B})\Gamma'' \to \mathfrak{A}_n; \quad \Gamma'\mathfrak{B}\Gamma'' \to \Theta}{\Gamma'((\mathfrak{A}_1 \supset (\mathfrak{A}_2 \supset \dots \supset (\mathfrak{A}_{n-1} \supset \mathfrak{A}_n) \cdots)) \supset \mathfrak{B})\Gamma'' \to \Theta},$$

where $\mathfrak{A}_1, \dots, \mathfrak{A}_{n-1}$ are arbitrary formulas, and $\mathfrak{A}_n$ is an arbitrary formula whose principal logical sign is different from the sign $\supset$.

In order, in searching for a derivation, to postpone as far as possible the examination of variants connected with the irreversible rules of inference of the calculus J_0, it is expedient to adjoin to the calculus J_0 a number of reversible rules based on well-known equivalences derivable in the constructive predicate calculus. The following rules may serve as examples of such admissible rules of inference:

$$\frac{\Gamma'(\mathfrak{A}' \supset (\mathfrak{A}'' \supset \mathfrak{B}))\Gamma'' \to \Theta}{\Gamma'((\mathfrak{A}' \& \mathfrak{A}'') \supset \mathfrak{B})\Gamma'' \to \Theta} \qquad \frac{\Gamma'\neg\mathfrak{A}\Gamma'' \to\ ; \quad \Gamma'\mathfrak{B}\Gamma'' \to}{\Gamma'(\mathfrak{A} \supset \mathfrak{B})\Gamma'' \to}$$

$$\frac{\Gamma'\neg\neg\mathfrak{A}\neg\mathfrak{B}\Gamma'' \to \Theta}{\Gamma'\neg(\mathfrak{A} \supset \mathfrak{B})\Gamma'' \to \Theta} \qquad \frac{\Gamma'\forall x\,\neg\mathfrak{A}\Gamma'' \to \Theta}{\Gamma'\neg\exists x\,\mathfrak{A}\Gamma'' \to \Theta}$$

In searching for a derivation it is also expedient, in suitable cases, to use Lemma 2 (for removing certain formulas from the antecedents of sequents).

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CITED LITERATURE

[1] V. A. Matulis, DAN, **147**, No. 5 (1962).
[2] S. C. Kleene, *Introduction to Metamathematics*, Moscow, 1957.
[3] G. Gentzen, Math. Zs., **39**, 176, 405 (1934–1935).
[4] N. N. Vorob' ev, Tr. Matem. inst. im. V. A. Steklova AN SSSR, **52**, 193 (1958).

Note: Figure translations are in progress. See original paper for figures.