SOME RELATIONS BETWEEN THE BEST POWER APPROXIMATION OF ANALYTIC FUNCTIONS AND THEIR EXPANSION COEFFICIENTS

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Abstract

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SOME RELATIONS BETWEEN THE BEST POWER APPROXIMATION OF ANALYTIC FUNCTIONS AND THEIR EXPANSION COEFFICIENTS

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The fundamental questions of the constructive theory of functions of one variable were studied by S. N. Bernstein. In particular, in ([1]) he found relations between the best uniform approximation of analytic functions and the coefficients of their expansion in a series.

In the present article limiting relations are established between the best power approximation of analytic functions of several variables and their expansion coefficients.

First we shall give some facts concerning the best power approximation of functions of several variables by means of generalized polynomials of several variables in the cube $D = D(|x_j| \leq 1,\ j = 1, \dots, k)$ with weight $\mu = \mu(x_1, \dots, x_k) > 0$.

Let $\mathfrak{M}^k$ denote a certain class of functions, and let $\{\varphi_\nu(x)\}$ be a system of functions given on the cube D, where $\nu = (m_1, \dots, m_k)$, m_j is a positive integer and $x = (x_1, \dots, x_k)$. With the aid of real numbers $a_{n\nu}$ we form the linear aggregate

$$S_n(x, a) = \sum_{|\nu| \leq n} a_{n\nu} \varphi_\nu. \tag{1}$$

Let $\|f\|_{p,n}^{(D,\mu)}$ be the best power approximation in the mean of a function $f \in \mathfrak{M}^k$, i.e. the lower bound of the norms

$$\|f - S_n(x, a)\|_{L_p(D,\mu)}, \qquad \text{where } 1 \leq p \leq \infty,$$

over the various possible combinations (1).

The nonnegative number

$$D_n(\mathfrak{M}^k, p) = \sup_{f \in \mathfrak{M}^k} \|f\|_{p,n}^{(D,\mu)}$$

will be called the best power approximation of the class $\mathfrak{M}^k$ by linear combinations (1).

In approximating functions of several variables by means of other functions we shall use the mean-square deviation; denoting it by $I(\cdot)$, we form the expression

$$w(\cdot) = cI^2(\cdot), \tag{2}$$

where c has a definite value depending only on the number of independent variables.

Lemma 1. Let $\{\varphi_\nu(x)\}$ be an arbitrary orthonormal system of functions square-integrable with positive weight μ in the cube D, and let f be a function given in the same cube D, also square-integrable with weight μ, and let n be a fixed natural number. Then $w(\cdot)$, among expressions of the form (1), attains its least value for $a_{n\nu} = c_{n\nu}$ ($\nu = 0, 1, ...$), where $c_{n\nu}$ are the generalized Fourier coefficients of the function f with respect to the given system with weight μ.

For the proof of the lemma, the method set forth in ([2]) is applied, where integral equations with a symmetric kernel are studied.

In ([3]) the constructive properties of bounded analytic functions in the elliptic polycylinder T_R^k, containing the k-dimensional cube D of the space C_z^k, were investigated. For the best power approximation of such functions by algebraic polynomials of several variables with weight μ, the limiting equality was established

$$\lim_{n\to\infty} \left(\|f\|_{p,n}^{(D,\mu)}\right)^{1/n} = s \qquad (s < 1). \tag{*}$$

Hence follows the existence of an analytic functional element which almost everywhere coincides with the function f being approximated, is continued continuously to the boundary of the elliptic polycylinder T_R^k ($R > 1$), and has singularities on its boundary.

Lemma 2. *If the function f is continuous in the cube D in the aggregate of its arguments and $\|f\|_{p,n}^{(D,1)}$ is the best power approximation of the function f by algebraic polynomials of several variables, then for $p = \infty$ the inequality*

$$|A_n| < c(k)\|f\|_{p,n}^{(D,1)}$$

holds, where A_n are the Fourier coefficients of the function f and $c(k)$ is a constant independent of n.

Proof. Since f is a continuous function in the cube D, there exists an algebraic polynomial in several variables P_n of degree n giving the best mean-square approximation to the function f, and minimizing the expression

$$w(f) = c\|f - P_n\|_{L_2(D,\mu)}$$

for each fixed function f.

This minimum is attained by the choice of the weight μ and of the coefficients of the polynomial P_n, and, on the basis of Lemma 1, is equal to

$$c(k) \sum_{|\nu| \geq n+1} A_\nu^2.$$

Obviously, for $p = \infty$,

$$\sum_{|\nu| \geq n+1} |A_\nu|^2 < c(k) \left(\|f\|_{p,n}^{(D,1)} \right)^2,$$

from which the assertion of the lemma follows.

Let us now denote by A_R^k the set of all functions analytic in the cube D. If the class $\mathfrak{M}^k$ coincides with the class of functions A_R^k, then the following holds.

Theorem 1. *Let the function $f \in A_R^k$, and let A_n be the coefficients of its expansion in a Fourier series, where n is the order of the coefficient. Then there exists an infinite set of values n for which the equality*

$$\limsup_{j\to\infty,\ n\geq j} \left(\|f\|_{p,n}^{(D,\mu)} \right)^{1/n} = \limsup_{j\to\infty,\ n\geq j} |A_n|^{1/n}.$$

is valid.

The theorem is proved with the aid of Lemmas 1, 2 and the inequality

$$\|f\|_{p,j}^{(D,\mu)} < c(s) s^j \qquad (s < 1),$$

obtained in ([3]).

Theorem 2. *Let f be an arbitrary function of the class A_R^k $(R > 1)$. Then, under the condition*

$$\lim_{R\to 1+0} \limsup_{j\to\infty,\ n\geq j} |A_n|^{1/n} = 1$$

the elliptic polycylinder of regularity of the function f contracts to the cube D, $f \in A^k_{1+0}$, and under the condition

$$\lim_{R\to\infty} \limsup_{j\to\infty,\ n\geq j} |A_n|^{1/n} = 0$$

the elliptic polycylinder of regularity of the function f, expanding, covers the whole space C^k_z and $f \in A^k_\infty$.

As is known, in the theory of entire functions one of the main problems is the study of the relation between the decrease of the coefficients and the growth of the maximum modulus of the function. To this end, for entire functions $f(z)$ of several complex variables, as well as for an entire function of one variable, the concepts of the order and type of a function are introduced and relations are established between these quantities and the coefficients of the expansion.

Let T^k_R be some arbitrary bounded complete k-circular domain of the space of variables $z_1, \dots, z_k$ with center at the origin, and let C^k_R be the boundary of the domain. For every function $f(z)$ analytic in the domain T^k_R and continuous in the closed domain $T^k_R + C^k_R$,

$$|f(z)| \leq \sup_{\xi\in C^k_R} |f(\xi)| = M_f(R), \qquad z \in T^k_R. \tag{3}$$

In the case of an unbounded domain T^k_R, inequality (3) will hold if one assumes, for example, that $|f(z)| \to 0$ as $\sum_{\tau=1}^k |z_\tau|^2 \to \infty$ for all $z \in T^k_R$.

Let now the function $f(z)$ be regular inside the bounded polycylinder $T^k_{R_1}$ ($1 < R_1 < R$). Consider the expansion of $f(z)$ in the cube D in a series in Chebyshev polynomials:

$$f(z) = \sum_{m_1,\dots,m_k=0}^{\infty} A_{m_1\dots m_k} T_{m_1}(z_1)\cdots T_{m_k}(z_k), \tag{4}$$

where

$$|A_{m_1\dots m_k}| \leq c_1(k) M_f(R_1) R_1^{-|m|}, \qquad |m| = \sum_{\tau=1}^k m_\tau, \tag{5}$$

the constant $c_1(k)$ does not depend on $|m|$. The series (4) converges absolutely and uniformly in the cube D. In ([3]) it is proved that

$$\|f\|^{(D,\mu)}_{p,n} < c(R_1)/R_1^n \qquad (n = 0, 1, \dots), \quad 1 \leq p \leq \infty. \tag{6}$$

Inequalities (5) and (6) are analogous to Cauchy' s inequality for the coefficients of the Taylor series of the function

$$f(z) = \sum_{m_1,\dots,m_k=0}^{\infty} a_{m_1\dots m_k} z_1^{m_1} \cdots z_k^{m_k}. \tag{7}$$

Theorem 3. If $f(z)$ is regular inside the elliptic polycylinder $T^k_{R_1}$ $(1 < R_1 < R)$ and on its boundary the inequality $|f(z)| < M_f(R_1)$ holds, then there exist functions $\tau_1(s)$, $\tau_2(s)$ such that the diameter $d(s)$ of the bounded complete k-circular domain of regularity T^k_r of the function $f(z)$ satisfies the relation

$$\tau_1(s) \leq d(s) \leq \tau_2(s),$$

where s is determined by the equality (*).

It is clear from the last theorem that the relation between the law of decrease of the best power approximation $\|f\|^{(D,\mu)}_{p,n}$ with unbounded increase of the degree of the polynomial and the law of growth of the maximum $M_f(R)$ of the modulus of the function $f = f(z_1, \dots, z_k)$ on the boundary of the elliptic polycylinder T^k_R as R increases can be studied by means of constructive function theory.

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Note: Figure translations are in progress. See original paper for figures.