DIFFERENTIAL PROPERTIES OF TRACES OF FUNCTIONS ON HYPERPLANES OF ARBITRARY DIMENSIONS

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Abstract

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DIFFERENTIAL PROPERTIES OF TRACES OF FUNCTIONS ON HYPERPLANES OF ARBITRARY DIMENSIONS

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Numerous works have been devoted to the study of the differential properties of traces of functions of many variables on hyperplanes of definite dimensions, depending on the membership of the function itself in one or another class of functions (see the survey article by S. M. Nikol'skii ([1])), in which these properties have been investigated fairly completely. The differential properties of traces of functions on hyperplanes of arbitrary dimensions, or on smooth manifolds of arbitrary dimensions, have been studied hardly at all. There are only a few papers ([2–6]) concerning this question, in which the Nikol' skii classes H_p^r and the partially generalized fractional spaces W_p^l of Sobolev are studied. In the present paper the differential properties are studied of traces of functions belonging to the classes

$$W_{p_0,p_1,p_2,\dots,p_s}^{(l_1,l_2,\dots,l_s)}(E^n) \quad \text{and} \quad B_{p_0,p_1,p_2,\dots,p_n\,\theta_1,\theta_2,\dots,\theta_n}^{(l_1,l_2,\dots,l_n)}(E^n)$$

on hyperplanes of arbitrary dimensions. In obtaining these results the author made use of certain results of the papers ([7],[8]).

In all that follows, unless otherwise specified, we shall use the following notation. Let E^n be n-dimensional Euclidean space. We represent each of its points $\bar{x}(x_1, x_2, \dots, x_n)$ in the form $\bar{x}(\bar{x}^1, \bar{x}^2, \dots, \bar{x}^s)$, where $\bar{x}^i(x_1^i, x_2^i, \dots, x_{n_i}^i)$, $i = 1, 2, \dots, s$, $\sum_1^s n_i = n$. Further, let E^{n_i} be the n_i-dimensional Euclidean space of the vectors $\bar{x}^i$; D_i an n_i-dimensional domain in E^{n_i}; ρ_i $(i = 0, 1, 2, \dots, s)$ positive numbers; $\rho_{i,j}$ the vector with coordinates $\rho_i, \rho_{i+1}, \dots, \rho_j$, where $0 \le i \le j \le s$; m_i $(i = 1, 2, \dots, k)$ natural numbers, with $1 \le k \le s$, $1 \le m_i \le n_i$ and $\sum_1^k m_i = m$; μ, ν nonnegative integers; E^{m_i} the m_i-dimensional Euclidean space of points $\bar{x}_1^i(x_1^i, x_2^i, \dots, x_{m_i}^i)$ or $\bar{y}_1^i(y_1^i, y_2^i, \dots, y_{m_i}^i)$; $E^{n_i - m_i}$ the $(n_i - m_i)$-dimensional Euclidean space of points $\bar{x}_2^i(x_{m_i+1}^i, \dots, x_{n_i}^i)$ or $\bar{y}_2^i(y_{m_i+1}^i, \dots, y_{n_i}^i)$; E^m the m-dimensional Euclidean space of points $\bar{x}_m(\bar{x}_1^1, \bar{x}_1^2, \dots, \bar{x}_1^k)$; E^{n-m} the $(n-m)$-dimensional Euclidean space of points $\bar{x}_{n-m}(\bar{x}_2^1, \bar{x}_2^2, \dots, \bar{x}_2^k, \bar{x}^{k+1}, \dots, \bar{x}^s)$; E^μ the μ-dimensional space of points $\bar{t}(t_1, t_2, \dots, t_\mu)$; E^ν the ν-dimensional space of points $\bar{z}(z_1, z_2, \dots, z_\nu)$; $h > 0$; $\chi_i > 0$ $(i = 0, 1, \dots, s)$;

$$r_{1,i}^2 = \sum_{j=1}^{m_i} (y_j^i - x_j^i)^2, \quad \text{if } 1 \le i \le k;$$

$$r_{2,i}^2 = \begin{cases} \sum\limits_{j=m_i+1}^{n_i} (y_j^i - x_j^i)^2, & \text{if } 1 \le i \le k, \\ \sum\limits_{j=1}^{n_i} (y_j^i - x_j^i)^2, & \text{if } k < i \le s. \end{cases}$$

$$r_i^2 = r_{1,i}^2 + r_{2,i}^2, \quad \text{if } 1 \le i \le k; \qquad r_i^2 = r_{2,i}^2, \quad \text{if } k < i \le s;$$

$$r^2 = \sum_{i=1}^{s} r_i^{2/\chi_i} + |\bar{t}|^{2/\chi_0}, \qquad \bar{r}^2 = \sum_{i=1}^{s} r_i^{2/\chi_i}; \qquad |\bar{t}|^2 = \sum_{j=1}^{\mu} t_j^2; \qquad |\bar{z}|^2 = \sum_{j=1}^{\nu} z_j^2;$$

$\Pi_\beta^\alpha(\bar{a})$ is the α-dimensional ball in E^α of radius β with center at the point $\bar{a} \in E^\alpha$.

In E^n consider the domain $D = D_1 \times D_2 \dots \times D_s$. Each function $f(\bar{x}) = f(\bar{x}^1, \bar{x}^2, \dots, \bar{x}^s)$ defined in D will be regarded as a function of the vector variables $\bar{x}^1, \bar{x}^2, \dots, \bar{x}^s$.

Denote by M the set of functions $f(\bar{x}^1, \bar{x}^2, \dots, \bar{x}^s)$, defined in D, for which the expression is bounded

$$\bar{A}_{(D_{1;s})}^{(\bar{\rho}_{1;s})}[f] \equiv \left(\int_{D_1} \left(\int_{D_2} \left(\cdots \left(\int_{D_s} f(\bar{x}^1, \bar{x}^2, \dots, \bar{x}^s)\, d\bar{x}^s \right)^{\rho_{s-1}/\rho_s} d\bar{x}^{s-1} \right)^{\rho_{s-2}/\rho_{s-1}} \cdots d\bar{x}^1 \right) \right)^{1/\rho_1}.$$

Let, further,

$$A_{(D_{0;s})}^{(\bar{\rho}_{0;s})}[F(\bar{x}, \bar{t})] \equiv \left\{ \int_{D_0} \left(A_{(D_{1;s})}^{(\bar{\rho}_{1;s})}[F(\bar{x}, \bar{t})] \right)^{\rho_0} d\bar{t} \right\}^{1/\rho_0},$$

where D_0 is a μ-dimensional domain in E^μ.

Theorem 1. Let $\rho_i \le q_1 \le q_2 < \infty$ $(i = 0, 1, \dots, s)$, $\beta > 0$,

$$\gamma > -\mu \frac{\rho_s}{\rho_s' \rho_0} \quad (\text{for } \mu = 0\ \gamma = 0); \qquad \frac{1}{\rho_s} + \frac{1}{\rho_s'} = 1; \qquad F(\bar{y}, \bar{t}) \in L_{\bar{\rho}_{s;0}}(E^{n+\mu}),$$

$$\lambda = \frac{\rho_s}{\rho_s'} \left(\sum_1^s \frac{n_i \chi_i}{\rho_i} + \frac{\mu \chi_0}{\rho_0} \right) + \left(\frac{1}{q_1} - \frac{1}{q_2} \right) \sum_1^k m_i \chi_i + \frac{1}{q_2} \sum_1^s n_i \chi_i;$$

then

$$\left\| \left(A^{(\bar\rho_{0;s})}_{(D_{0;s})} \left[\frac{|\bar t|^{\gamma} F(y,\bar t)}{\left(\sqrt{r^2+H^2}\right)^{\lambda+\gamma\chi_0+\alpha}} \right] \right)^{\rho_s} \right\|_{L(q_1,q_2)(E^n)} \le$$

$$\le \begin{cases} ch^{\beta}\|F\|_{L_{\bar\rho_{s;0}}(E^{n+\mu})}, & \text{if } \alpha=-\beta,\ H=0, \\ ch^{-\beta}\|F\|_{L_{\bar\rho_{s;0}}(E^{n+\mu})}, & \text{if } \alpha=\beta,\ 0<H\le h, \\ c\left\| \dfrac{F|\bar t|^{\gamma}}{\left(\sqrt{|\bar t|^{2/\chi_0}+H^2}\right)^{\gamma\chi_0}} \right\|_{L_{\bar\rho_{s;0}}(E^n\times\Pi^{\mu}_{h\chi_0}(\bar 0))}, & \text{if } 0\le H\le h,\ \gamma\ge 0, \end{cases}$$

where

$$D_{0;s}=\prod_1^s D_i, \qquad D_0=\Pi^{\mu}_{h\chi_0}(\bar 0),$$

$$D_i=\begin{cases} \Pi^{m_i}_{h\chi_i}(\bar x^{\,i}_1)\times\Pi^{n_i-m_i}_{h\chi_i}(\bar x^{\,i}_2), & \text{if } 1\le i\le k, \\ \Pi^{n_i}_{h\chi_i}(\bar x^{\,i}), & \text{if } k<i\le s. \end{cases}$$

Theorem 2. If $1\le\rho_i\le q_1\le q_2<\infty$; $\rho_0<\theta\le\sigma<\infty$; $\chi_i>0$; $\varepsilon\ge 0$; $\beta>0$; $\alpha\ge 0$; $h>0$; $\lambda_i=\chi_i/\chi$ $(i=0,1,\dots,s)$; $\gamma>-\mu\rho_s/\rho_s'\rho_0$;

$$\lambda=\frac{\rho_s}{\rho_s'}\left(\sum_1^s\frac{n_i\chi_i}{\rho_i}+\frac{\mu\chi_0}{\rho_0}\right)+\left(\frac{1}{q_1}-\frac{1}{q_2}\right)\sum_1^k m_i\chi_i+\frac{1}{q_2}\sum_1^s n_i\chi_i;$$

$$\|\varphi\|_{\mathfrak{L}^{(\alpha)}_{\rho_s;1}(E^n,\Pi^{\mu}_{h\chi_0}(\bar 0))}\equiv\left[\int_{\Pi^{\mu}_{h\chi_0}(\bar 0)}\frac{d\bar t}{|\bar t|^{\mu+\alpha\theta}}\left(A^{(\bar\rho_{1;s})}_{(E^n)}\left[|\varphi(\bar y,\bar t)|^{\rho_s}\right]\right)^{\theta}\right]^{1/\theta}<\infty,$$

then:

I. $$\left\{\int_{\Pi^{\nu}_{h\chi}(\bar 0)}\frac{d\bar z}{|\bar z|^{\nu+\sigma\beta}}\left\|\left(A^{(\bar\rho_{0;s})}_{(D_{0;s})}\left[\frac{|\bar t|^{-\mu/\rho_0-(\alpha-\gamma)}\varphi(\bar y,\bar t)}{r^{\lambda+\gamma\chi_0-\beta\chi-\varepsilon}}\right]\right)^{\rho_s}\right\|^{\sigma}_{L(q_1,q_2)(E^n)}\right\}^{1/\sigma}\le$$

$$\le ch^{\varepsilon}\|\varphi\|_{\mathfrak{L}^{(\alpha)}_{\rho_s;1}(E^n\times\Pi^{\mu}_{h\chi_0}(\bar 0))}.$$

II.

$$\left\{\int_{\Pi^{\nu}_{h\chi(\bar 0)}}\frac{d\bar z}{|\bar z|^{\nu-\sigma\beta}}\left\|\left(A^{(\rho_0;s)}_{(D_0;s)}\left[\frac{|t|^{-\mu/\rho_0-(\alpha-\gamma)}\varphi(\bar y,\bar t)}{\left(\sqrt{r^2+|\bar z|^2/\chi_0}\right)^{\lambda+\gamma\chi_0+\beta\chi-\varepsilon}}\right]\right)^{\rho_s}\right\|^{\sigma}_{L_{(q_1,q_2)}(E^n)}\right\}^{1/\sigma}\le$$

$$\le ch^{\varepsilon}\|\varphi\|_{L^{(\alpha)}_{\rho_s;1}(E^n\times\Pi^{\mu}_{h\chi_0}(\bar 0))},$$

where

$$D_0 = \Pi^{\mu}_{|\bar{z}|\lambda_0}(\bar{0}),$$

$$D_i = \begin{cases} \Pi^{m_i}_{|\bar{z}|\lambda_i}(\bar{x}^i_1) \times \Pi^{n_i - m_i}_{|\bar{z}|\lambda_i}(\bar{x}^i_2), & \text{if } 1 \le i \le k, \\ \Pi^{n_i}_{|\bar{z}|\lambda_i}(\bar{x}^i), & \text{if } k < i \le s. \end{cases}$$

Theorem 3. If $F(\bar{y}, \bar{t}) \in L_p(E^{n+\mu})$, $1 < p < q_1 < q_2 < \infty$, and

$$\lambda = \frac{1}{p'} \left(\sum_1^s n_i \chi_i + \mu \chi_0 \right) + \left(\frac{1}{q_1} - \frac{1}{q_2} \right) \sum_1^k m_i \chi_i + \frac{1}{q_2} \sum_1^s n_i \chi_i,$$

then

$$\left\| \int_{E^\mu} d\bar{t} \int_{E^n} F(\bar{y}, \bar{t})\, r^{-\lambda}\, d\bar{y} \right\|_{L_{(q_1, q_2)}(E^n)} \le c \|F\|_{L_p(E^{n+\mu})}.$$

Theorem 4. Let

$$f \in W^{(l_1, \dots, l_s)}_{p_0, p_1, \dots, p_s}(E^n)$$

and let $\nu^{(i)}_j$ $(i = 1, 2, \dots, s,\ j = 1, 2, \dots, n)$ be nonnegative integers satisfying the conditions

$$\sum_1^{n_i} \nu^{(i)}_j = \nu^{(i)}, \qquad \sum_1^s \nu^{(i)} = \nu; \qquad 1 \le p_i \le q_1 \le q_2 \le \infty \quad (i = 0, 1, \dots, s);$$

$$\varepsilon = 1 - \sum_1^s \frac{n_i}{l_i p_i} - \sum_1^s \chi_j \nu^{(j)} + \left(\frac{1}{q_1} - \frac{1}{q_2} \right) \sum_1^k m_i \chi_i + \frac{1}{q_2} \sum_1^s n_i \chi_i \ge 0;$$

then

$$\|D^\nu_x f\|_{L_{(q_1, q_2)}(E^n)} \le \begin{cases} c \left(h^{-\delta} \|f\|_{L_{p_0}(E^n)} + h^\varepsilon \|f\|_{L^{(l_1, l_2, \dots, l_s)}_{p_1, p_2, \dots, p_s}(E^n)} \right), \\ \quad \text{if } \varepsilon > 0 \text{ and } 1 \le p_i \le q_1 \le q_2 < \infty\ (i = 0, 1, \dots, s), \\ c \left(h^{-\delta} \|f\|_{L_{p_0}(E^n)} + \|f\|_{L^{(l_1, l_2, \dots, l_s)}_{p_1, p_2, \dots, p_s}(E^n)} \right), \\ \quad \text{if } \varepsilon = 0,\ 1 < p_i < q_1 < q_2 < \infty\ (i = 1, 2, \dots, s), \end{cases}$$

where h is arbitrary positive, and

$$\delta = 1 - \varepsilon + \sum_1^s \frac{n_i}{l_i} \left(\frac{1}{p_0} - \frac{1}{p_i} \right).$$

Theorem 5. If

$$f \in B^{(l_1, \dots, l_n)}_{p_0, p_1, \dots, p_n;\, \theta_1, \dots, \theta_n}(E^n), \qquad 1 \le p_i \le \theta_i < \infty$$

and

$$1-\sum_1^n \frac{1}{l_i p_i}-\sum_1^n \nu_i\chi_i+\left(\frac{1}{q_1}-\frac{1}{q_2}\right)\sum_1^m \chi_i+\frac{1}{q_2}\sum_1^n \chi_i=\varepsilon\geq 0,$$

then

$$D_x^\nu f\|_{L_{(q_1,q_2)}(E^n)}\leq\begin{cases} c\left(h^{-\delta}\|f\|_{L_{p_0}(E^n}+h^\varepsilon\|f\|_{\Omega^{(l_1,\ldots,l_n)}_{p_1,\ldots,p_n;\theta_1,\ldots,\theta_n}(E^n)}\right), \\ \quad\text{if } \varepsilon>0,\ 1\leq p_i\leq q_1\leq q_2<\infty\ (i=1,2,\ldots,n), \\ c\left(h^{-\delta}\|f\|_{L_{p_0}(E^n)}+\|f\|_{\Omega^{(l_1,\ldots,l_n)}_{p_1,\ldots,p_n}(E^n)}\right), \\ \quad\text{if } \varepsilon=0,\ \theta_i=p_i,\ 1<p_i<q_1<q_2<\infty,\ i=1,2,\ldots,n,\end{cases}$$

where

$$\delta=1-\varepsilon-\sum_1^n\frac{1}{l_i}\left(\frac{1}{p_0}-\frac{1}{p_i}\right).$$

Theorem 6. If $f\in B^{(l_1,\ldots,l_n)}_{p_0,p_1,\ldots,p_n;\theta_1,\ldots,\theta_n}(E^n)$, where $1\leq p_i\leq\theta_i<\infty$, l_i are nonnegative integers, and the numbers q_1,q_2,σ,ρ_k satisfy the conditions $1\leq p_i\leq q_1\leq q_2<\infty$; $\rho_k\chi_k\leq\varepsilon$ $(k,i=1,2,\ldots,n)$,

$$\varepsilon=1-\sum_1^n\frac{1}{l_i p_i}-\sum_1^n\nu_i\chi_i+\left(\frac{1}{q_1}-\frac{1}{q_2}\right)\sum_1^m\chi_i+\frac{1}{q_2}\sum_1^n\chi_i>0,$$

then

I. $D^\nu_{\bar x}f(\bar x)\in B^{(\rho_1,\rho_2,\ldots,\rho_n)}_{\sigma;(q_1,q_2)}(E^n)$, and the inequalities

$$\|D^\nu_{\bar x}f(\bar x)\|_{B^{(\rho_k)}_{\sigma;(q_1,q_2)}(E^n)}\leq c\left[h^{-\delta_k}\|f\|_{L_{p_0}(E^n)}+h^{\varepsilon-\rho_k\chi_k}\left(1+h^{\rho_k\chi_k}\right)\|f\|_{B^{(l_1,\ldots,l_n)}_{p_1,\ldots,p_n;\theta_1,\ldots,\theta_n}(E^n)}\right].$$

II. If $\rho_k\chi_k\leq\varepsilon$, then $D^\nu_{\bar x}f(\bar x)\in W^{(\rho_1,\ldots,\rho_n)}_{(q_1,q_2)}(E^n)$, and

$$\|D^\nu_{\bar x}f(\bar x)\|_{W^{(\rho_k)}_{(q_1,q_2)}(E^n)}\leq c\left[h^{-\delta_k}\|f\|_{L_{p_0}(E^n)}+h^{\varepsilon-\rho_k\chi_k}\left(1+h^{\rho_k\chi_k}\right)\|f\|_{B^{(l_1,\ldots,l_n)}_{p_1,\ldots,p_n;\theta_1,\ldots,\theta_n}(E^n)}\right]$$

(if $\varepsilon=\rho_k\chi_k$, then $1<p_i<q_1<q_2<\infty$; $i=1,2,\ldots,n$), where

$$\delta_k=1-(\varepsilon-\rho_k\chi_k)+\sum_1^n\frac{1}{l_i}\left(\frac{1}{p_0}-\frac{1}{p_i}\right),\qquad k=1,2,\ldots,n.$$

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Note: Figure translations are in progress. See original paper for figures.