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Abstract

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MATHEMATICS

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SOME BOUNDARY-VALUE PROBLEMS FOR SYSTEMS OF SECOND-ORDER EQUATIONS OF ELLIPTIC TYPE IN THE PLANE

(Presented by Academician M. A. Lavrent' ev on 10 VIII 1964)

1. In this paper the following problem is considered:
In a finite $(m+1)$-connected plane domain D with smooth boundary $\Gamma = \Gamma_0 + \dots + \Gamma_m$, it is required to find a regular solution of the elliptic system

$$\mathcal{L}_{xy}(u) = \sum_{i+j\leq 2} A_{ij} \frac{\partial^{i+j} u}{\partial x^i \partial y^j} = h, \tag{1}$$

belonging to the class $C'_\alpha(\overline{D})$ and satisfying the boundary condition

$$B_{10}u_x + B_{01}u_y + B_{00}u = f \quad \text{on } \Gamma, \tag{2}$$

where $u = (u_1, \dots, u_n)$ is the unknown vector, $h = (h_1, \dots, h_n)$ and $f = (f_1, \dots, \dots, f_n)$ are given real vectors respectively in $\overline{D}$ and on Γ; A_{ij} and B_{ij} are real square matrices of order n, given respectively in D and on Γ.

It is assumed that the matrices A_{ij} and B_{ij} belong respectively to the classes $C_\alpha^{i+j}(\overline{D})$ and $C_\alpha^{i+j}(\Gamma)$, while f and h belong respectively to the classes $C_\alpha(\Gamma)$ and $C_\alpha(\overline{D})$.

For a sufficiently broad class of elliptic systems, problem (1)–(2) was studied in works ([1–4]). In work ([5]) a sufficient condition for the Noether property of problem (1)–(2) was obtained (the condition of Ya. B. Lopatinskii). Elliptic systems satisfying this condition form a broader class than the systems considered in works ([1–4]). For a simply connected domain D, problem (1)–(2) was considered in work ([6]). In that work problem (1)–(2) is reduced to an integral equation not always equivalent to problem (1)–(2); however, under the fulfillment of the indicated condition, the Noether property of this problem is proved and a formula for the index is obtained.

In the present paper a method is given for reducing problem (1)–(2) to an equivalent one-dimensional singular integral equation. In addition, a condition

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is indicated (condition (11)) under which this equation is of normal type and problem (1)–(2) is Noetherian.

2. Along with equation (1), consider the equation adjoint to (1),

$$\mathcal{L}^*_{xy}(u) \equiv \sum_{i+j\le 2} (-1)^{i+j} \frac{\partial^{i+j}}{\partial x^i \partial y^j}(uA_{ij}) = 0. \tag{3}$$

Let

$$K(x,y,\lambda) = \sum_{i+j=2} A_{ij}\lambda^j \equiv \|a_{ij}(x,y,\lambda)\|,$$

$$\Delta(x,y,\lambda) = \|\Delta_{ij}\|(\det K(x,y,\lambda))^{-1},$$

where Δ_{ij} $(i,j=1,\dots,n)$ is the algebraic cofactor of the element a_{ji} in the determinant $\det K(x,y,\lambda)$.

Consider the matrix

$$v(x,y,\xi,\eta) = \frac{1}{2\pi^2}\operatorname{Re}\int_\gamma \Delta(x,y,\lambda)\ln((\xi-x)+\lambda(\eta-y))\,d\lambda, \tag{4}$$

where γ is a contour in the half-plane $\operatorname{Im}\lambda > 0$ enclosing all roots λ of the polynomial $\det K(x,y,\lambda)$ that lie in this half-plane. The elements of the matrix (4) are single-valued functions of the variables x, y, ξ, η $((x,y)\ne(\xi,\eta))$.

Denote

$$\widetilde{\Omega}(x,y,\xi,\eta) \equiv \int_{(x_k,y_k)}^{(x,y)} \Omega(x,y,\xi_k(s),\eta_k(s))\,ds \qquad ((x,y)\in D, \quad (\xi,\eta)\in\Gamma_k),$$

where

$$\Omega(x,y,\xi,\eta) = \frac{\partial}{\partial\xi}(vA_{20}(\xi,\eta))\cos(\nu,\xi) + \frac{\partial}{\partial\xi}(vA_{11}(\xi,\eta))\cos(\nu,\eta) + \frac{\partial}{\partial\eta}(vA_{02}(\xi,\eta))\cos(\nu,\eta) + v(A_{10}\cos(\nu,\xi) + A_{01}\cos(\nu,\eta));$$

$\xi=\xi_k(s)$, $\eta=\eta_k(s)$ are parametric equations of the contour Γ_k; s is the arc length measured from the fixed point $(x_k,y_k)\in\Gamma_k$ in the positive direction; ν is the inward normal to the curve Γ at the point (ξ,η).

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Transforming the expression

$$\iint_D (w\mathcal{L}_{\xi\eta}(u) - \mathcal{L}^*_{\xi\eta}(w)u)\, d\xi\, d\eta$$

by the known Green formula and assuming that $u(\xi,\eta)$ is a solution of equation (1), and $w = v(x,y,\xi,\eta)$, where (x,y) is a fixed point in the domain D, (ξ,η) is the point of integration, we obtain

$$\begin{aligned} u(x,y) + \iint_D \mathcal{L}^*_{\xi\eta}(v)\, u(\xi,\eta)\, d\xi\, d\eta = \iint_D vh(\xi,\eta)\, d\xi\, d\eta + \int_\Gamma G_1 \frac{\partial u}{\partial \xi}\, ds \\ + \int_\Gamma G_2 \frac{\partial u}{\partial \eta}\, ds + \sum_{k=0}^{m} b_k(x,y) u(x_k, y_k), \end{aligned} \tag{5}$$

where

$$G_1 = -vA_{20}(\xi,\eta)\cos(\nu,\xi) - \widetilde{\Omega}(x,y,\xi,\eta)\cos(\nu,\eta),$$

$$G_2 = -v(A_{11}(\xi,\eta)\cos(\nu,\xi) + A_{02}(\xi,\eta)\cos(\nu,\eta)) + \widetilde{\Omega}(x,y,\xi,\eta)\cos(\nu,\xi),$$

$$b_k(x,y) = \int_{\Gamma_k} \Omega(x,y,\xi,\eta)\, ds.$$

Here and below (ξ,η) will be the point of integration. We note that $\mathcal{L}^*_{\xi\eta}(v)$ may have, at the points $(\xi,\eta) = (x,y)$, a pole of order no higher than the first.

We represent the matrix $\mathcal{L}^*_{\xi\eta}(v)$ as a sum of square matrices of order n in the following form:

$$\mathcal{L}^*_{\xi\eta}(v) = \|\alpha_{ij}(x,y,\xi,\eta)\| + \sum_{k=1}^{l} \gamma_k(x,y)\|\beta_{ij}(\xi,\eta)\|, \tag{6}$$

where

$$\max_{1\le i\le n} \iint_D \sum_{j=1}^{n} |\alpha_{ij}(x,y,\xi,\eta)|\, d\xi\, d\eta < q, \qquad q = \text{const}, \qquad q < 1.$$

Identity (5) can be written in the form

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$$u = \iint_D M_1 h(\xi,\eta)\,d\xi\,d\eta + \int_\Gamma \Omega_1 u_\xi\,ds + \int_\Gamma \Omega_2 u_\eta\,ds + \sum_{k=1}^{l} N_k(x,y)c_k + \sum_{k=0}^{m} \omega_k(x,y)u(x_k,y_k), \tag{7}$$

$$M_1 = M(v(x,y,\xi,\eta)),\quad \Omega_i = M(G_i(x,y,\xi,\eta)),\quad N_k = M(\gamma_k(x,y)),$$

$$\omega_k = M(b_k(x,y)),\quad M(\psi(x,y,\xi,\eta)) \equiv \psi(x,y,\xi,\eta) + \iint_D \widetilde{K}(x,y,\xi,\eta)\psi(x,y,\xi,\eta)\,d\xi\,d\eta,$$

$\widetilde{K}(x,y,\xi,\eta)$ is the resolvent of the Fredholm equation with kernel $\|a_{ij}(x,y,\xi,\eta)\|$; $c_k = (c_{k1},\dots,c_{kn})$ is a constant vector,

$$c_{ki} = -\sum_{j=1}^{n} \iint_D \beta_{ij}^{(k)}(\xi,\eta)u_j(\xi,\eta)\,d\xi\,d\eta \quad (k=1,\dots,l;\ i=1,\dots,n).$$

On the basis of (7), it is natural to seek the solution of problem (1)–(2) in the form

$$u = \iint_D M_1 g(\xi,\eta)\,d\xi\,d\eta + \int_\Gamma \Omega_1 \psi(\xi,\eta)\,ds + \int_\Gamma \Omega_2 \psi(\xi,\eta)\,ds +$$

$$+ \sum_{k=1}^{l} N_k(x,y)c_k + \sum_{k=0}^{m} \omega_k(x,y)d_k, \tag{8}$$

where $c_1,\dots,c_l,d_0,\dots,d_m$ are constant n-dimensional vectors; $g(\xi,\eta)$ is an n-dimensional vector function of class $C_\alpha(\overline{D})$; $\varphi(\xi,\eta)$ and $\psi(\xi,\eta)$ are likewise n-dimensional vector functions of class $C_\alpha(\Gamma)$, satisfying the equations

$$B_{10}(x,y)\varphi(x,y) + B_{01}(x,y)\psi(x,y) + B_{00}d_k +$$

$$+B_{00}(x,y)\int_{(x_k,y_k)}^{(x,y)} (\cos(\nu,\eta)\varphi(\xi,\eta) - \cos(\nu,\xi)\psi(\xi,\eta))\,ds = f \quad \text{on } \Gamma_k \tag{9}$$

$$(k=0,1,\dots,m)$$

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and the conditions

$$\int\limits_{\Gamma_k} (\cos(\nu,\eta)\varphi(\xi,\eta) - \cos(\nu,\xi)\psi(\xi,\eta))\, ds = 0 \quad (k = 0, 1, \dots, m). \tag{10}$$

Substituting (8) into equation (1) and into the boundary condition (2), we obtain integral equations with respect to g, φ, and ψ. These equations, together with equation (9) and condition (10), are equivalent to our problem, and they can be solved in the following way. The integral equations obtained by substituting (8) into (1) form a system of Fredholm integral equations of the second kind with respect to $g(x, y)$. We solve this system of integral equations with respect to $g(x, y)$, regarding φ and ψ as known. Substituting the value found for $g(x, y)$ into the remaining integral equations, we obtain one-dimensional singular integral equations with respect to φ and ψ.

Thus, the resulting system of singular integral equations is a system of normal type if the condition

$$\det \begin{Vmatrix} A & B \\ B_{10} & B_{01} \end{Vmatrix} \neq 0 \quad \text{on } \Gamma, \tag{11}$$

is satisfied, where

$$A(x,y) = \frac{1}{2\pi} \int\limits_{\gamma} (B_{10} + \lambda B_{01}) \Delta(x,y,\lambda)(\cos(\nu,y) - \lambda\cos(\nu,y))^{-2} \times$$

$$\times [A_{20}(-\lambda\cos^2(\nu,x) + 2\cos(\nu,x)\cos(\nu,y)) +$$

$$+ A_{11}\cos^2(\nu,y) + \lambda A_{02}\cos^2(\nu,y)]\, d\lambda,$$

$$B(x,y) = -\frac{1}{2\pi} \int_{\gamma} (B_{10} + \lambda B_{01}) \Delta(x,y,\lambda)(\cos(\nu,y) -$$
$$-\lambda\cos(\nu,x))^{-2}\, [A_{20}\cos^2(\nu,x) + \lambda A_{11}\cos^2(\nu,x) +$$
$$+ A_{02}(2\lambda\cos(\nu,x)\cos(\nu,y) - \cos^2(\nu,y))]\, d\lambda.$$

Hence, by the known theorems of the theory of singular integral equations (see (7)), it follows

Theorem. *If condition (11) is satisfied, the homogeneous problem (1)–(2) has a finite number of linearly independent solutions, and for the solvability of the nonhomogeneous problem (1)–(2) it is necessary and sufficient that the functions $h(x, y)$ and $f(s)$ satisfy a finite number of conditions of the form*

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$$\iint_D h(x,y)w_i(x,y)\,dx\,dy + \int_\Gamma f(s)\alpha_i(s)\,ds = 0. \tag{12}$$

Condition (11), in its form of notation, differs from the condition of Ya. B. Lopatinskii, but for many boundary-value problems they are equivalent. In the general case the question of the relation between these conditions remains open.

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