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1965

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Abstract

Full Text

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ON SOME EXTREMAL PROBLEMS IN THE THEORY OF POLYNOMIAL OPERATORS

(Presented by Academician S. N. Bernstein, November 10, 1964)

1°. Let us introduce notation. Π_n is the set of all trigonometric polynomials of order $\leq n$; L_1 is the set of all summable 2π-periodic functions; E is a linear normed function space possessing the following properties: 1) the elements of E are functions from L_1; 2) if $f \in E$, then the shifted function $f_t(x) = f(x+t)$, for any $-\infty < t < \infty$, also belongs to E, and $\|f_t\| = \|f\|$; 3) E contains the set of all trigonometric polynomials. The most important special cases of the space E are: the space C of all continuous 2π-periodic functions, and the space L_r of all 2π-periodic functions summable to the r-th power. Put

$$\sigma_n(f,x) = \int_0^{2\pi} f(x+t)\Phi(t)\,dt,$$

where

$$\Phi(t) = \frac{r_0}{2} + \sum_{k=1}^{n} r_k \sin(kt+\alpha_k). \tag{1}$$

Denote by $\Omega_n^\Phi(E)$ the set of all linear operations U_n from E into E possessing the property that $U_n(t_n) = \sigma_n(t_n)$ if $t_n \in \Pi_n$. The set $\Omega_{n,n}^\Phi(E)$ consists of all linear operations $U_{n,n}$ from E into E for which the conditions are fulfilled: 1) for every $f \in E$, $U_{n,n}(f) \in \Pi_n$; 2) if $t_n \in \Pi_n$, then $U_{n,n}(t_n) = \sigma_n(t_n)$. It is obvious that $\Omega_{n,n}^\Phi(E) \subset \Omega_n^\Phi(E)$. There exist operations belonging to $\Omega_n^\Phi(E)$ but not belonging to $\Omega_{n,n}^\Phi(E)$. Introduce the numbers

$$\rho_n(E) = \rho_n^\Phi(E) = \inf_{U_n \in \Omega_n^\Phi(E)} \|U_n\|; \qquad \rho_{n,n}(E) = \rho_{n,n}^\Phi(E) = \inf_{U_{n,n} \in \Omega_{n,n}^\Phi(E)} \|U_{n,n}\|.$$

It is clear that

$$\rho_{n,n}(E) \geq \rho_n(E). \tag{2}$$

The ratio $\rho_{n,n}(E) : \rho_n(E)$ depends essentially on the space E (2). In L_2 the following theorem holds.

Theorem 1. The equalities

$$\rho^{\Phi}_{n,n}(L_2) = \rho^{\Phi}_{n}(L_2) = \pi \max_{j=0,1,\dots,n} r_j \tag{3}$$

hold.

Let us outline the proof. It is easy to see that

$$\left\| U_n\left(\frac{\cos kx}{\|\cos kx\|_E}\right)\right\| = \pi r_k, \qquad k = 0,1,2,\dots,n,$$

where $\|\cos kx\|_E$ is the norm of $\cos kx$ in the metric of E. Therefore

$$\rho_n(E) \geq \pi r_{j_0}, \qquad r_{j_0} = \max_{j=0,1,2,\dots,n} r_j. \tag{4}$$

With the aid of Parseval' s equality it is easy to obtain that

$$\|\sigma_n\|_{L_2} \leq \pi r_{j_0}.$$

Consequently, since $\sigma_n \in \Omega^{\Phi}_{n,n}(E)$, we have

$$\rho_{n,n}(E) \leq \pi r_{j_0}. \tag{5}$$

From (2) and (4), (5), (3) follows.

Of particular interest is the case when

$$\Phi(t) = \frac{1}{\pi} D^{(k)}_n(t), \tag{6}$$

where $D_n(t)$ is the Dirichlet kernel and $D^k_n(t)$ is the derivative of order k. In this case $r_{j_0} = n^k/\pi$, and therefore equality (3) takes the form

$$\rho_{n,n}(L_2) = \rho_n(L_2) = n^k. \tag{7}$$

The validity of equality (7) was pointed out to the author by A. N. Kolmogorov.

2°. In the case $E = \widetilde{C}$ or $E_1 = \widetilde{L}_1$, the study of the ratio $\rho_{n,n}(E) : \rho_n(E)$ is considerably more difficult. In [1] it was established that if $\Phi(t)$ is defined according to (6), then

$$\lim_{n\to\infty}\left(\frac{\rho_{n,n}(\widetilde{C})}{\rho_n(\widetilde{C})} : \frac{4}{\pi^2}\ln n\right) = 1. \tag{8}$$

(8) also remains valid when $\widetilde{C}$ is replaced by $\widetilde{L}_1$. The question arises of studying $\rho_{n,n}(E) : \rho_n(E)$ in the case of an arbitrary kernel $\Phi(t)$ of the form (1).

Theorem 2. Let $E = \widetilde{C}$ or $E = \widetilde{L}_1$. Suppose that the kernel $\Phi(t)$ satisfies the conditions $r_0 = 0$ and

$$\Phi_1(t) = r_n + 2\sum_{k=1}^{n-1} r_n \cos[(n-k)t + \alpha_n - \alpha_k] \geq 0, \qquad -\infty < t < \infty. \tag{9}$$

Then the equality holds

$$\rho^{\Phi}_{n,n}(E)/\rho^{\Phi}_n(E) = \int_0^{2\pi} |\Phi(t)|\, dt \;/\; \pi r_n.$$

In the course of the proof the following lemma plays an important role:

Lemma. Suppose that the kernel $\Phi(t)$ satisfies the conditions of Theorem 2. Then, for an arbitrary space of type E, the equality holds

$$\rho^{\Phi}_n(E) = \pi r_n. \tag{10}$$

We outline the proof. Put

$$\overline{U}(f, x) = \overline{U}(f, g, x) = \int_0^{2\pi} f(x+t) g(nt + \alpha_n) \Phi_1(t)\, dt, \tag{11}$$

where $\Phi_1(t)$ is defined according to (9), and $g(t)$ is an arbitrary 2π-periodic continuous function whose Fourier expansion begins with $\sin t$. It is not difficult to verify that $\overline{U} \in \Omega^{\Phi}_n(E)$. In view of (11) we have

$$\|\overline{U}(f)\| \leq \|f\| \int_0^{2\pi} |g(nt + \alpha_n)|\, |\Phi_1(t)|\, dt. \tag{12}$$

The legitimacy of passing to the norm under the integral sign in the case $E = C$ or $E = \widetilde{L}_1$ is obvious. In the general case it is easy to justify. Following the arguments of F. Riesz ([2]), put

$$g(t) = \sin t - r^2 \sin 3t + r^4 \sin 5t - \cdots = \frac{(1-r^2)\sin t}{1 + 2r^2 \cos 2t + r^4}, \quad 0 < r < 1.$$

Since

$$\int_0^{2\pi} |g(t)|\, dt = \frac{4}{r} \operatorname{arc\,tg} r$$

and $\Phi_1(t) \geq 0$, it follows from (12) that $\|\overline{U}\| \leq 4r_n r^{-1} \operatorname{arc\,tg} r$. Therefore

$$\rho_n^{\Phi}(E) \leq 4r_n \operatorname{arc\,tg} r/r.$$

Letting $r \to 1$, we obtain

$$\rho_n^{\Phi}(E) \leq \pi r_n. \tag{13}$$

On the other hand, by virtue of (4) one always has $\rho_n^{\Phi}(E) \geq \pi r_n$. From this and (13), (10) follows.

Remark. Equality (8) is a special case of Theorem 2.

3°. Let $L = L_{[a,b]}$ be the set of all summable on the segment $[a,b]$ functions $f(x)$ with $\|f\| = \int_a^b |f|\, dx$. Consider some subspace Σ of the space L and a linear operator U from L into Σ. Put that $\{\chi_i\}_{i=1}^p$ are eigenfunctions of the operator U, $U(\chi_k) = \mu_k \chi_k$, $k = 1, 2, \dots, p$. We shall assume that $|\chi_k(x)| \leq \Delta$, $k = 1, 2, \dots, p$, where Δ does not depend on x or k.

Theorem 3. *If (in L) there exist a sequence of functionals $\{\psi_k\}_{k=1}^p$ and a sequence of positive numbers $\{a_k\}_{k=1}^p$ such that for any $f \in \Sigma$ the inequality*

$$\sum_{k=1}^p a_k |\psi_k(f)| \leq C \int_a^b |f|\, dx,$$

where C is a constant, holds, then

$$\|U\| \geq \sum_{k=1}^p a_k \mu_k \psi_k(\chi_k^2) \,/\, C\Delta(b-a).$$

A special case of this theorem is

Theorem 4. *Let U_n be a linear operator from $\widetilde{L}_1$ into Π_n, and let the polynomials $\{t_k^{(n)}\}_{k=1}^m$, $m \leq n$, be its fixed points, where $t_k^{(n)}$ is a polynomial of degree k. Then, if the polynomials $\{t_k^{(n)}\}_{k=1}^n$ are uniformly bounded in the aggregate, then*

$$\|U_n\| \geq C_1 \ln \frac{n}{n-m+1}.$$

From this estimate the theorem follows:

Theorem 5. *Let* $\overline{\lim}_{n\to\infty}\frac{m_n}{n}=1$, *and let* $\{U_n\}_{n=1}^{\infty}$ *be a sequence of linear operators from* $\widetilde{L}_1$ *into* $\widetilde{L}_1$, *where* U_n, $n=1,2,\ldots$, *has the properties: a) for any* $f\in\widetilde{L}_1$, $U_n(f)\in\Pi_n$; *b) there exists a sequence of polynomials* $\{t_k^{(n)}\}_{k=1}^{m_n}$, $m_n\le n$, $n=1,2,\ldots$ ($t_k^{(n)}$ *is a polynomial of degree* k), *uniformly bounded in the aggregate, such that* $U_n(t_k^{(n)})=t_k^{(n)}$, $k=1,2,\ldots,m_n$, $n=1,2$.

Then for some $f\in\widetilde{L}_1$ the equality

$$\overline{\lim_{n\to\infty}}\|U_n(f)-f\|_{\widetilde{L}_1}=\infty.$$

This theorem is a strengthening of the Lozinskii-Kharshiladze theorem ([3]), since in the latter it is required that any polynomial $t_n\in\Pi_n$ be a fixed point of the operator U_n. The conditions of Theorem 5 are satisfied, for example, by the polynomials

$$t_k^{(n)}(x)=\sin(kx+a_k^{(n)}),$$

$$k=1,2,\ldots,m_n;\qquad n=1,2,\ldots\quad(m_n\le n).$$

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Received
2 X 1964

REFERENCES

[1] D. L. Berman, DAN, 138, No. 4 (1961). [2] F. Riesz, C. R., 158 (1914). [3] I. P. Natanson, *Constructive Function Theory*, 1949.

Note: Figure translations are in progress. See original paper for figures.