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Abstract

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MATHEMATICS

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BOUNDARY VALUES OF FUNCTIONS FROM WEIGHTED CLASSES

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In the present paper we consider weighted classes of functions defined in the half-space $x_n > 0$ of points $x = (x_1, \dots, x_n)$ of the n-dimensional space E_n. The properties of the traces of these functions on the $(n-1)$-dimensional subspace E_{n-1}, defined by the equation $x_n = 0$, are studied.

Let r be a natural number; $1 < p < \infty$; E^+ the upper half-space of points with $x_n > 0$, and $\varphi = \varphi(\rho)$ $(\rho = \sqrt{x_1^2 + \dots + x_n^2})$ a positive function (weight) defined on E^+; let $f = f(x)$ be a function defined on E^+ together with its generalized derivatives up to order r inclusive. By definition, the function f belongs to the class $L^{(r)}_{p,\varphi}(E^+)$ or $W^{(r)}_{p,\varphi}(E^+)$, if for it the finite norm

$$\|f\|_{L^{(r)}_{p,\varphi}(E^+)} = \sum_{|k|=r} \left\| \frac{f^{(k)}(x)}{\varphi(\rho)} \right\|_{L_p(E^+)}$$

or

$$\|f\|_{W^{(r)}_{p,\varphi}(E^+)} = \|f\|_{L^{(r)}_{p,\varphi}(E^+)} + \|f\|_{L_p(\omega)}$$

has meaning, where $k = (k_1, \dots, k_n)$ is an integer vector $(k_i \geq 0,\ i = 1, \dots, n)$, $|k| = k_1 + k_2 + \dots + k_n$, ω is the intersection of E^+ with the unit ball in E_n centered at the origin, and

$$\|f\|_{L_p(\omega)} = \left\{ \int_\omega |f|^p \, dx \right\}^{1/p}, \qquad f^{(k)}(x) = \frac{\partial^{|k|} f(x)}{\partial x_1^{k_1} \cdots \partial x_n^{k_n}}.$$

For $\varphi = 1$, the classes $L^{(r)}_{p,\varphi}$ and $W^{(r)}_{p,\varphi}$ become the well-known classes $L^{(r)}_p$ and $W^{(r)}_p$ of S. L. Sobolev (^{1}).

Let l be a positive non-integer number and $l = \bar{l} + \alpha$, where $\bar{l}$ is an integer and $0 < \alpha < 1$. By definition, a function $\psi = \psi(x_1, \dots, x_{n-1})$ belongs to the class $L_{p,\varphi}^{(l)}(E_{n-1})$ or $W_{p,\varphi}^{(l)}(E_{n-1})$, if it is given on E_{n-1} together with its generalized derivatives up to order $\bar{l}$ inclusive and has the finite norm

$$\|\psi\|_{L_{p,\varphi}^{(l)}(E_{n-1})} = \sum_{|k|=\bar{l}} \sum_{i=1}^{n-1} \left\{ \int_0^\infty \frac{dh}{h^{1+\alpha p}} \left\| \frac{\Delta_i(\psi^{(k)}, h)}{\varphi(\bar{\rho} + h)} \right\|_{L_p(E_{n-1})}^p \right\}^{1/p}$$

or

$$\|\psi\|_{W_{p,\varphi}^{(l)}(E_{n-1})} = \|\psi\|_{L_p(\omega_*)} + \|\psi\|_{L_{p,\varphi}^{(l)}(E_{n-1})},$$

where $\Delta_i(\psi, h)$ $(i = 1, \dots, n-1)$ denotes the first difference of the function $\psi = \psi(x_1, \dots, x_{n-1})$ with step h in the variable x_i; $\bar{\rho} = |x_1| + \dots + |x_{n-1}|$, and $\omega_* = \omega E_{n-1}$. Let the weight function $\varphi = \varphi(t)$ on $[0, \infty)$ be continuous, positive, nonincreasing, and satisfy the inequality $\varphi(2t) \le c\varphi(t)$

(for sufficiently large t), where c is a positive constant independent of t. Then the following theorems hold.

Theorem 1. If the function $f \in L_{p,\varphi}^{(r)}(E^+)$ or $W_{p,\varphi}^{(r)}(E^+)$, then for it the boundary function

$$\psi = \psi(x_1, \dots, x_{n-1}) = f\big|_{E_{n-1}}$$

has meaning, and respectively the embeddings* hold

$$L_{p,\varphi}^{(r)}(E^+) \to L_{p,\varphi}^{(r-1/p)}(E_{n-1}), \qquad W_{p,\varphi}^{(r)}(E^+) \to W_{p,\varphi}^{(r-1/p)}(E_{n-1}). \tag{1}$$

Theorem 2. If the function $\psi \in L_{p,\varphi}^{(r-1/p)}(E_{n-1})$, $W_{p,\varphi}^{(r-1/p)}(E_{n-1})$, then there exists a function $f \in L_{p,\varphi}^{(r)}(E^+)$, $W_{p,\varphi}^{(r)}(E^+)$, defined on E^+, such that

$$f\big|_{E_{n-1}} = \psi$$

and the embeddings hold

$$L_{p,\varphi}^{(r-1/p)}(E_{n-1}) \to L_{p,\varphi}^{(r)}(E^+), \qquad W_{p,\varphi}^{(r-1/p)}(E_{n-1}) \to W_{p,\varphi}^{(r)}(E^+). \tag{2}$$

Theorem 3. Suppose that on E_{n-1} a system of functions is given

$$\psi_i \in L_{p,\varphi}^{(r-i-1/p)}(E_{n-1}),\ W_{p,\varphi}^{(r-i-1/p)}(E_{n-1}) \qquad (i = 0, 1, \dots, r-1).$$

Then there exists a function $f \in L_{p,\varphi}^{(r)}(E^+), W_{p,\varphi}^{(r)}(E^+)$, defined on E^+, such that

$$\partial^i f/\partial x_n^i\big|_{E_{n-1}} = \psi_i \qquad (i = 0, 1, \dots, r-1)$$

and the inequality is fulfilled

$$\|f\|_{L_{p,\varphi}^{(r)}(E^+)} \le c \sum_{i=0}^{r-1} \|\psi_i\|_{L_{p,\varphi}^{(r-i-1/p)}(E_{n-1})}$$

and, respectively, the analogous inequality in which everywhere L is to be replaced by W.

In the present paper the investigations of L. D. Kudryavtsev (^{2}) are developed. The classes considered here are generalizations of the weighted classes $L_{p,\alpha}^{(r)}$, $W_{p,\alpha}^{(r)}$ introduced by him ($\varphi = (1+\rho)^\alpha$), for which he obtained direct and inverse embedding theorems. But our embedding theorems differ from the corresponding theorems of L. D. Kudryavtsev. Along with the class $L_{p,\alpha}^{(r)}(E^+)$ he also considers the class $L_{p,\alpha}^{(r)}(E_1^+)$ of functions defined on the strip $E_1^+\{x : 0 < x_n \le 1\}$. Moreover, he introduces on E_{n-1} fractional classes $\bar{L}_{p,\alpha}^{(r)}(E_{n-1})$ and proves that

$$L_{p,\alpha}^{(r)}(E^+) \to \bar{L}_{p,\alpha}^{(r-1/p)}(E_{n-1}) \to L_{p,\alpha}^{(r)}(E_1^+).$$

In our case, however, the inverse embedding theorem completely reverses the direct theorem (see (1) and (2)). We have also succeeded in showing that the fractional classes of L. D. Kudryavtsev $\bar{L}_{p,\alpha}^{(r-1/p)}(E_{n-1})$ are not equivalent to $L_{p,\varphi}^{(r-1/p)}(E_{n-1})$ (for $\varphi = (1+\rho)^\alpha,\ \alpha > 0$).

Embeddings for unweighted spaces W_p^r were studied in the works (^{1},^{3},^{4},$^{12-15}$), while embeddings of the spaces $L_p^{(r)}$, i.e. spaces in which the norm of the elements does not include the norm of the function itself in L_p, were studied in ($^{2-4}$,^{15}).

* If a function f defined on E^+ belongs to the normed space Λ_1, and its trace $\psi = f|_{E_{n-1}}$ belongs to the normed space Λ_2 and $\|\psi\|_{\Lambda_2} \le c_1 \|f\|_{\Lambda_1}$, then, as usual, we write $\Lambda_1 \to \Lambda_2$. Conversely, $\Lambda_2 \to \Lambda_1$ means that there exists on E^+ a function $f \in \Lambda_1$ such that $f|_{E_{n-1}} = \psi$ and $\|f\|_{\Lambda_1} \le c_2 \|\psi\|_{\Lambda_2}$, where c_1 and c_2 are constants independent of f, ψ.

Theorem 4. Let the weight function be $\varphi = \rho^{\alpha_0}\lambda(\rho)$ ($\alpha_0 = (n-p)/p$), $1 < p < \infty$, and suppose that $\lambda(\rho)$ satisfies the condition

$$\int_1^\infty \frac{dz}{z\chi(z)} < \infty, \qquad \chi(z) = \left\{ \min_{1 \le t < \infty} \left[\frac{\lambda(zt)}{\lambda(t)} \right]^p \right\}^{1/p}.$$

Then the embedding holds

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$$W^{(r)}_{p,\varphi}(E^+) \to W^{(r-k)}_{p,(1+\rho)^k\varphi}(E^+), \qquad k = 0, 1, \dots, r.$$

The last theorem is analogous to the first assertion of Theorem 1 of paper (^{5}); the classes $W^{(r)}_{\rho,\varphi}$ considered there were defined somewhat differently than in the present paper.

Consider the differential equation

$$L(u) = \sum_{|k|,\,|l| \le r} (-1)^{|l|} \frac{\partial^l}{\partial x^l} (a_{kl}(x) u^{(k)}(x)) = F \tag{3}$$

in the class $\mathfrak{M}$ of functions f belonging to $L^{(r)}_{2,\varphi}(E^+)$ and satisfying the boundary conditions

$$\partial^i f / \partial x_n^i \big|_{E_{n-1}} = \psi_i \qquad (i = 0, 1, \dots, r-1),$$

where $\psi_i \in L^{(r-i-1/2)}_{2,\varphi}(E_{n-1})$ are prescribed functions. It is assumed that

$$a_{kl}(x) = a_{lk}(x), \qquad |a_{kl}(x)| \le M^2 / [(1+\rho)^{r-\min(|k|,|l|)} \varphi(\rho)]^2,$$

$$\sum_{|k|,\,|l| \le r} a_{kl} \xi_k \xi_l \ge \frac{\lambda}{[\varphi(\rho)]^2} \sum_{|k|=r} \xi_k^2,$$

where $\lambda > 0$ does not depend on x, ξ_k, ξ_l. The function F has the property that the norm

$$\sigma_F = \|F\|_{L_2^{(r)}(E^+)} = \sup_{\substack{\|v\|_{L^{(r)}_{2,\varphi}(E^+)} \le 1 \\ v \in \mathfrak{M}_0}} |(F, v)|, \tag{4}$$

is finite, where

$$(F, v) = \int_{E^+} F \cdot v \, dx$$

and $\mathfrak{M}_0$ is the class of functions $f \in L^{(r)}_{2,\varphi}(E^+)$ having zero boundary functions $\psi_i = 0$ $(i = 0, 1, \dots, r-1)$. In particular, if

$$\int_{E^+} (1+\rho)^{2r} \varphi^2 F^2 \, dx < \infty,$$

then condition (4) is satisfied.

Relying essentially on Theorems 1, 3, and 4, we prove by the variational method the following theorems.

Theorem 5. In the class $\mathfrak{M}$ there exists, and moreover is unique, a generalized solution u of equation (3).

Theorem 6. For the generalized solution U of the boundary-value problem under consideration, the estimate

$$\|U\|_{W_{2,\varphi}^{(r)}(E^+)} \leq c\left\{\sum_{i=0}^{r-1}\|\psi_i\|_{W_{2,\varphi}^{(r-i-1/2)}(E_{n-1})} + \sigma_F\right\},$$

holds, where c is a constant independent of the boundary functions ψ_i ($i = 0, 1, \dots, r-1$) and of the quantity σ_F.

If one requires the functions $a_{kl}(x)$ and $F(x)$ to be sufficiently smooth, then from known results (see, for example, ($^{6-9}$)) it follows that the solution U has continuous partial derivatives on E^+ up to order $2r$ inclusive and becomes a classical solution of equation (3).

Theorem 7. *The classical solution u of equation (3) in the class $\mathfrak{M}$ is unique.*

The proof of the last theorem is based essentially on the fact that it is shown that the class $\mathfrak{M}_{00}$ is everywhere dense in the class $\mathfrak{M}_0$ in the sense of the metric

$$D(f) = \int_{E^+}\sum_{|k|=r}\left[\frac{f^{(k)}(x)}{\varphi(\rho)}\right]^2 dx,$$

where $\mathfrak{M}_{00}$ is the class of finite functions $f \in \mathfrak{M}_0$.

We note that these investigations develop the works of L. D. Kudryavtsev (^{10}, ^{11}), in which the variational method for solving the first boundary-value problem in the case of an unbounded domain was considered for self-adjoint elliptic equations of the second order.

In conclusion I express my deep gratitude to L. D. Kudryavtsev for posing the problem and for his constant attention.

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