MATHEMATICS

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Abstract

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MATHEMATICS

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ON EXPANSION IN EIGENFUNCTIONS OF A NON-SELF-ADJOINT BOUNDARY-VALUE PROBLEM FOR A DIFFERENTIAL EQUATION WITH A SINGULARITY AT ZERO

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1. Consider the operator generated by the differential equation

$$-y'' + \left\{ \frac{l(l+1)}{x^2} + q(x) \right\} y = \lambda^2 y \tag{1}$$

and the boundary condition

$$y(0) = 0, \tag{2}$$

where l is an integer; $q(x)$ is a complex-valued, locally summable function on the half-axis $[0, \infty)$. The results of the present work are analogous to the results of V. A. Marchenko (^{1}), M. A. Naimark (^{2}), and V. E. Lyantse (^{3,4}) for problem (1)–(2) with $l = 0$. To obtain these results we first study in greater detail the kernel of the transformation considered earlier by V. V. Stashevskaya (^{5}) and V. Ya. Volk (^{6}).

2. Let $\varphi_l(x, \lambda)$ be a solution of equation (1) such that

$$\lim_{x \to 0} \frac{\varphi_l(x, \lambda)}{x^{l+1}} = \frac{1}{2^{l+1/2}\Gamma(l + 3/2)}. \tag{3}$$

It is then known (^{5,6}) that there exist functions $K(x, t)$ and $H(x, t)$ such that

$$\varphi_l(x, \lambda) = \frac{\sqrt{x}}{\lambda^{l+1/2}} J_{l+1/2}(\lambda x) + \int_0^x K(x, t) \frac{\sqrt{t}}{\lambda^{l+1/2}} J_{l+1/2}(\lambda t)\, dt, \tag{4}$$

$$\frac{\sqrt{x}}{\lambda^{l+1/2}} J_{l+1/2}(\lambda x) = \varphi_l(x, \lambda) + \int_0^x H(x, t) \varphi_l(t, \lambda)\, dt. \tag{5}$$

Theorem 1. If $q(x)$ has l locally summable derivatives, then

$$K(x,t) = \frac{t^{l+1}}{x^l} K_1(x,t), \qquad H(x,t) = \frac{t^{l+1}}{x^l} H_1(x,t), \tag{6}$$

where the functions $K_1(x,t)$ and $H_1(x,t)$ have, with respect to both variables, $l+1$ locally summable derivatives.*

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* In the works (^{5,6}) it is proved only that, for every t,

$$\int_0^x |K(x,t)|^2 \, dt < \infty, \qquad \int_0^x |H(x,t)|^2 \, dt < \infty.$$

3. Definition. The space of all even entire functions of finite degree, summable with the weight $\lambda^{2(l+1)}$ on the real axis, is called the space W_l. A sequence $\{F_n(\lambda)\}$ in the space W_l converges to zero if the degrees σ_n of all the functions $F_n(\lambda)$ are uniformly bounded and

$$\lim_{n\to\infty} \int_0^\infty \lambda^{2(l+1)} |F_n(\lambda)| \, d\lambda = 0.$$

Theorem 2. *If the function $q(x)$ has l locally summable derivatives, then to the boundary-value problem (1)–(2) there corresponds a certain continuous linear functional R on the topological space W_l such that if $f(x) \in K$, $g(x) \in K$, where K is the set of all finite functions in $L_2(0,\infty)$, and*

$$F(\lambda) = \int_0^\infty f(x) \varphi_l(x,\lambda) \, dx, \tag{7}$$

$$G(\lambda) = \int_0^\infty g(x) \varphi_l(x,\lambda) \, dx, \tag{8}$$

then

$$(R, F(\lambda) G(\lambda)) = \int_0^\infty f(x) g(x) \, dx. \tag{9}$$

The continuous linear functional R is called the **spectral function** of problem (1)–(2).

We outline the proof of Theorem 2. Let $\Phi(\lambda) \in W_l$. Define the functional R on W_l as follows:

$$\left(R, \frac{\Gamma(l + {}^3/_2)}{2^{l + {}^3/_2}} \Phi(\lambda) \right) = \lim_{x\to 0} \frac{\varphi(x)}{x^{l+1}} + \int_0^\infty H_1(x,0) \frac{\varphi(x)}{x^l} \, dx, \tag{10}$$

where $H_1(x,0)$ is from Theorem 1,

$$\varphi(x) = \int_0^\infty \lambda^{l+{}^{3}/{}_{2}} \Phi(\lambda) J_{l+1/2}(\lambda x)\, dx. \tag{11}$$

It is obvious that R is a continuous linear functional on the space W_l. Now we must prove that this functional satisfies the requirements of the theorem. For this, note that if $f_0(x)$ and $g_0(x)$ belong to the space K, and $F_0(\lambda)$ and $G_0(\lambda)$ are their

$$\frac{\sqrt{x}}{\lambda^{l+1/2}} J_{l+1/2}(\lambda x)$$

Fourier transforms, then, as is seen from formula (10),

$$(R, F_0(\lambda) G_0(\lambda)) = \int_0^\infty f_0(x) g_0(x)\, dx + \int_0^\infty \int_0^\infty F(x,y) f_0(x) g_0(y)\, dx\, dy, \tag{12}$$

where

$$F(x,y) = \begin{cases} H(x,y) + \displaystyle\int_0^y H(x,t) H(y,t)\, dt, & \text{for } y < x, \\ H(y,x) + \displaystyle\int_0^x H(x,t) H(y,t)\, dt, & \text{for } y > x. \end{cases}$$

It follows from (4) that if $F(\lambda)$ and $G(\lambda)$ are defined by formulas (7) and (8), respectively, then

$$F(\lambda) = \int_0^\infty \left[f(x) + \int_x^\infty K(t,x) f(t)\, dt \right] \frac{\sqrt{x}}{\lambda^{l+1/2}} J_{l+1/2}(\lambda x)\, dx,$$

$$G(\lambda) = \int_0^\infty \left[g(x) + \int_x^\infty K(t,x) g(t)\, dt \right] \frac{\sqrt{x}}{\lambda^{l+1/2}} J_{l+1/2}(\lambda x)\, dx.$$

Now, replacing in (12) the functions $f_0(x)$ and $g_0(x)$ by

$$f(x) + \int_x^\infty K(t,x) f(t)\, dt, \qquad g(x) + \int_x^\infty K(t,x) g(t)\, dt$$

and using the relation between the functions $H(x,t)$, $K(x,t)$, and $F(x,t)$, one can fully prove the theorem.

From formulas (10) and (11) there immediately follows

Corollary. If $\Phi(\lambda) \in W_l$ and

$$\Phi(\lambda) = \int_0^\infty \varphi(x)\varphi_l(x, \lambda)\, dx, \tag{13}$$

then

$$(R, \Phi(\lambda)) = \frac{2^{l+1/2}}{\Gamma(l + 3/2)} \lim_{x\to 0} \frac{\varphi(x)}{x^{l+1}}. \tag{14}$$

In conclusion of this section let us note that if $q(z)$ is an entire function, then Theorem 2 remains valid also for nonintegral $l \geq \frac{1}{2}$.

4. Until now we have imposed no restrictions on the behavior at infinity of the function $q(x)$. Now suppose that the integral

$$\int_0^\infty e^{\varepsilon x} |q(x)|\, dx \tag{15}$$

converges for some $\varepsilon > 0$. Then one can show that for any $l \geq \frac{1}{2}$ the problem (1)–(2) has a finite number of eigenvalues, and its continuous spectrum coincides with the positive half-axis. We denote by $f_l(x, \lambda)$ the solution of equation (1) which, as $x \to \infty$, has the asymptotic form $e^{i\lambda x}$. Let

$$A(\lambda) = (-1)^l e^{-i\frac{l}{4}\pi} \frac{\Gamma(-l + 1/2)}{2^{l+1/2}} \lim_{x\to 0} \frac{f_l(x, \lambda)}{(\lambda x)^l}.$$

It is not difficult to show that $A(\lambda)$ is a holomorphic function in the half-plane $\operatorname{Im}\lambda > -\varepsilon/2$, and its zeros in the upper half-plane coincide with the eigenvalues of problem (1)–(2).

Theorem 3. *Let the function $q(x)$ satisfy condition (15), the function $A(\lambda)$ not vanish on the real axis, and the zeros in the upper half-plane be simple. Then the kernel of the resolvent of the operator (1)–(2) has the form*

$$R_\lambda(x, y) = \sum_{k=1}^{m} \frac{\varphi_l(x, \lambda_k)\psi_l(y, \lambda_k)}{(\lambda_k^2 - \lambda^2) \displaystyle\int_0^\infty [\varphi_l(x, \lambda_k)]^2\, dx} + \int_0^\infty \frac{\varphi_l(x, s)\varphi_l(y, s) s^{2(l+1)}}{(s^2 - \lambda^2)A(s)A(-s)}\, ds,$$

where $\lambda_1, \dots, \lambda_m$ are the eigenvalues of problem (1)–(2), and if λ does not belong to the spectrum of the operator (1)–(2), then

$$\int_0^\infty |R_\lambda(x, y)|^2\, dy < \infty.$$

From the asymptotic formula $A(\lambda) = 1+O(1/\lambda)$ as $\lambda \to +\infty$, Theorem 4 follows immediately.

Theorem 4. If $f(x) \in L_2(0,\infty)$ and

$$F(\lambda) = \int_0^\infty f(x)\varphi_1(x,\lambda)\,dx$$

for real λ, and

$$F(\lambda_n) = \int_0^\infty f(x)\varphi_1(x,\lambda_n)\,dx,$$

then

$$f(x) = \sum_{k=1}^{m} \frac{F(\lambda_n)}{\displaystyle\int_0^\infty [\varphi_1(x,\lambda_n)]^2\,dx}\,\varphi_1(x,\lambda_n) + \int_0^\infty \frac{F(s)\varphi_1(x,s)}{A(s)A(-s)} s^{2(l+1)}\,ds,$$

where the last integral converges in the metric of $L_2(0,\infty)$.

Here we have assumed that $A(\lambda)$ has no zeros on the real axis. Otherwise, so-called spectral singularities appear (see (3)), and the formulations of the corresponding results become more complicated.

5. In this section we give the solution of the inverse problem for equation (1).

First we introduce notation. The set of

$$\frac{\sqrt{x}}{\lambda^{l+1/2}}\,J_{l+1/2}(\lambda x)$$

transforms of functions from K will be denoted by W_l^2.

Theorem 5. In order that a certain linear functional R on the space W_l be a spectral function of a problem of type (1)–(2), for integral l and l-times locally summable function $q(x)$, it is necessary and sufficient that the following conditions be fulfilled:

1) if for all $F(\lambda) \in W_l^2$ and for fixed $G(\lambda) \in W_l^2$

$$(R, F(\lambda)G(\lambda)) = 0,$$

then $G(\lambda) \equiv 0$;

2) the function

$$\left(R,\int_0^x \frac{\sqrt{s}}{\lambda^{l+1/2}} J_{l+1/2}(\lambda s)\,ds \int_0^y \frac{\sqrt{t}}{\lambda^{l+1/2}} J_{l+1/2}(\lambda t)\,dt\right)$$

has, in both variables, $l+3$ locally summable derivatives.

In conclusion, we note that V. V. Stashevskaya (^{5}) studied the inverse problem for the self-adjoint equation of type (1) for integral l, and later M. G. Krein (^{7}) solved this problem completely, reducing its solution to the inverse problem for an equation without singularities. For self-adjoint equations of type (1), we succeeded in completely solving the inverse problem for all $l \geq \frac{1}{2}$ and locally summable $q(x)$. We shall not dwell on these results here.

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Note: Figure translations are in progress. See original paper for figures.

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