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Abstract

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ON MULTIPLE COMPLETENESS AND CONVERGENCE OF MULTIPLE EXPANSIONS WITH RESPECT TO THE SYSTEM OF EIGENVECTORS AND ASSOCIATED VECTORS OF AN OPERATOR PENCIL

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In the present note some criteria are established for multiple completeness of the system of eigenvectors and associated vectors of an operator pencil and for summability of multiple expansions with respect to this system by the Abel method in Banach (Sec. 1) and Hilbert (Sec. 2) spaces. In Sec. 3 some theorems are given on convergence of multiple expansions with respect to the indicated system in a Hilbert space. In what follows, without further qualification, the terminology and notation of the papers ([1], [2]) are used.

1. All operators considered in this section act in some Banach space $\mathfrak{B}$.

Theorem 1. *Let the operator* $H \in \mathfrak{S}_p$ *for some* $p > 0$ *and have a purely real spectrum, and let*

$$\|(H - \lambda I)^{-1}\| \leq C|\operatorname{Im}\lambda|^{-1}$$

($\operatorname{Im}\lambda \neq 0$)*. If* $\mathfrak{R}(H) = \mathfrak{B}$ *and* $T_\nu \in \mathfrak{S}_\infty$ $(\nu = 0, 1, \dots, n-1)$*, then the system of eigenvectors and associated vectors of each of the pencils*

$$L(\lambda) = I - T_0 - \lambda T_1 H - \dots - \lambda^{n-1} T_{n-1} H - \lambda^{nH}, \tag{1}$$

$$M(\lambda) = I - T_0 - \lambda H T_1 - \dots - \lambda^{n-1} H T_{n-1} - \lambda^{nH} \tag{2}$$

is n-fold complete in $\mathfrak{B}$*.*

This theorem is a somewhat weakened analogue of a theorem obtained by M. V. Keldysh ([3]) for the case of a Hilbert space (in ([3]) it is assumed that $T_\nu \in \mathfrak{R}$, $\nu = 1, 2, \dots, n-1$).

Let us outline the proof of Theorem 1, considering, for definiteness, the pencil $L(\lambda)$. In this case the proof of the theorem is reduced to the proof of completeness of the system of eigenvectors and associated vectors of the linear pencil $\tilde{I} - \widetilde{S} - \lambda\widetilde{G}$ in the space $\widetilde{\mathfrak{B}}$. Here $\widetilde{\mathfrak{B}}$ is the product of n copies of the space

$\mathfrak{B}$; $\tilde{I}$ is the identity operator in $\widetilde{\mathfrak{B}}$; the operators $\widetilde{S}$ and $\widetilde{G}$ are given in $\widetilde{\mathfrak{B}}$ by the matrices $\widetilde{S} = (S_{jk})_1^n$, $\widetilde{G} = (G_{jk})_1^n$, where $S_{11} = T_0$, $S_{1k} = T_{k-1}H$ $(k > 1)$, $S_{jk} = 0$ $(j > 1)$, $G_{1n} = H$, $G_{j,j-1} = I$ $(j > 1)$, $G_{jk} = 0$ $(k - j \neq -1, n)$ (see (4)). If the operator $I - T_0$ is invertible, then the proof reduces to establishing completeness of the system of root vectors of the operator $\widetilde{A} = (\tilde{I} - \widetilde{S})^{-1}\widetilde{G}$. One can show that $\widetilde{A}^n = (\tilde{I} + \widetilde{T})\widetilde{H}$, where $\widetilde{T} \in \mathfrak{S}_\infty$, and $\widetilde{H} = (\delta_{jk}H)_1^n$, and therefore for the operator $\widetilde{A}^n$ the conditions of Theorem 1 from (1)* are satisfied.

Remark 1. Let $n(r, L)$ (respectively $n(r, H)$) be the number of characteristic numbers of the pencil $L(\lambda)$ (respectively of the operator H), counted with their multiplicities, lying in the disk $|\lambda| < r$. It is not difficult to show that if, under the conditions of Theorem 1, the spectrum of H lies on the positive half-axis, then

$$\varlimsup_{r\to\infty} n(r, L)/n(r, H) \geq n$$

and

$$\varliminf_{r\to\infty} n(r, L)/n(r, H) \leq n.$$

If, moreover, there exists a nondecreasing function $\varphi(r)$ $(r > 0)$ such that

$$\lim_{r\to\infty} n(r, H)/\varphi(r) = 1$$

and such that for all sufficiently large r and $s > r$ it is fulfilled— the inequality $\varphi(s)/\varphi(r) \leqslant (s/r)^\gamma$ $(\gamma > 0)$, then* $\lim_{r\to\infty} n(r, L)/n(r, H) = n$. The assertions stated are also valid for the pencil $\widetilde{M}(\lambda)$.

Modifying the method of proof of Theorem 1, one can show that it remains valid also in the case $T_\nu \in \mathfrak{S}$ $(\nu = 1, 2, \dots, n-1)$, and also establish the following generalization of a theorem of M. V. Keldysh (3) (see also (6)).

Theorem 2. Let the operator $H \in \mathfrak{S}_p$ $(p > 0)$ and let the spectrum of H lie on the rays $l_\nu = \{z : \arg z = \nu\pi/n\}$ $(\nu = 0, 1, \dots, 2n-1)$, and suppose that

$$\|(H - \lambda I)^{-1}\| \leqslant C[\rho(\lambda, l_\nu)]^{-1}$$

$$(0 < |\arg\lambda - \nu\pi/n| \leqslant \pi/2n;\ \nu = 0, 1, \dots, 2n-1).$$

If $\mathfrak{R}(H) = \mathfrak{B}$ and $T_\nu \in \mathfrak{S}_\infty$ $(\nu = 0, 1, \dots, n-1)$, then the system of eigenvectors and associated vectors of each of the pencils

$$I - T_0 - \gamma T_1 H - \dots - \lambda^{n-1}T_{n-1}H^{n-1} - \lambda^n H^n, \qquad (3)$$

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* We note that in Theorems 1, 3, 7 and in the lemma from (1) the condition $T \in \mathfrak{S}_\infty^{(0)}$ may be replaced by the condition $T \in \mathfrak{S}_\infty$.

$$I - T_0 - \lambda H T_1 - \ldots - \lambda^{n-1} H^{n-1} T_{n-1} - \lambda^n H^n \tag{4}$$

is n-fold complete in $\mathfrak{B}$.

Remark 2. The condition $H \in \mathfrak{S}_p$ may be replaced in Theorems 1 and 2 by any of the following conditions: a) for some m the operator H^m is nuclear; b) $\sum_i |\lambda_i(H)|^p < \infty$; c) $\sum_i d_i^p(H) < \infty$, where $d_n(H)$ is the n-width (^{7}) of the set into which the operator H maps the unit sphere of the space $\mathfrak{B}$.

In the work of V. B. Lidskii (^{8}) the notion of Abel summability of series in the system of root vectors of a linear operator was introduced. We shall need below a generalization of this notion to operator pencils.

Let $\{\varphi_j^{(\nu)}\}_{j=1}^{\infty}$ $(\nu = 0, 1, \ldots, n-1)$ be a canonical set of systems of eigenvectors and associated vectors of the pencil $M(\lambda)$. To any set of n vectors $f_\nu \in \mathfrak{B}$ $(\nu = 0, 1, \ldots, n-1)$ there are uniquely assigned series in the sequences $\{\varphi_j^{(\nu)}\}_{j=1}^{\infty}$ $(\nu = 0, 1, \ldots, n-1)$ with identical coefficients

$$f_\nu \sim \sum_{j=1}^{\infty} c_j \varphi_j^{(\nu)} \quad (\nu = 0, 1, \ldots, n-1). \tag{5}$$

Let $\alpha > 0$ be given and

$$P_m^\alpha(\zeta^{-1}, t) = \frac{\exp(t\zeta^{-\alpha})}{m!} \frac{d^m}{d\zeta^m} \exp(-t\zeta^{-\alpha}) \quad (m = 0, 1, \ldots).$$

If $\varphi_j^{(0)}, \varphi_{j+1}^{(0)}, \ldots, \varphi_{j+k}^{(0)}$ $(k \geq 0)$ is a chain of eigenvectors and associated vectors corresponding to the characteristic number λ_j, and the multiplicity of $\varphi_j^{(0)}$ is $k+1$, then we put

$$c_{j+i}(t) = \exp(-t\lambda_j^{n\alpha}) \sum_{m=0}^{k-i} P_m^\alpha(\lambda_j^n, t) c_{j+i+m} \quad (i = 0, 1, \ldots, k).$$

If there exists a subsequence of the natural series $\{m_k\}_0^\infty$ $(m_0 = 1)$ such that for all $t > 0$ the series

$$u_\nu(t) = \sum_{k=1}^{\infty} \left(\sum_{j=m_{k-1}}^{m_k - 1} c_j(t) \varphi_j^{(\nu)} \right) \quad (\nu = 0, 1, \ldots, n-1)$$

converge and $\lim_{t \to +0} u_\nu(t) = f_\nu$ $(\nu = 0, 1, \ldots, n-1)$, then we shall say that for the set f_ν $(\nu = 0, 1, \ldots, n-1)$ the n-fold expansions (5) in the system of eigenvectors and associated vectors of the pencil $M(\lambda)$ are summable by the Abel method of order α.

* This equality is proved on the basis of Tauberian theorem of B. I. Korenblyum (^{5}).

Theorem 3. Let $H \in \mathfrak{S}_\infty$ be an operator whose spectrum is contained in the angle

$$\Lambda = \{z : \ |\arg z| \le \gamma\pi/2\} \qquad (\gamma < 2),$$

and suppose that

$$\|(H - \lambda I)^{-1}\| \le C[\rho(\lambda, \Lambda)]^{-1} \qquad (\lambda \notin \Lambda).$$

If $\mathfrak{R}(H) = \mathfrak{B}$, $T_\nu \in \mathfrak{S}_\infty$ $(\nu = 0, 1, \dots, n-1)$, and at least one of the two conditions is fulfilled: a) $H \in \mathfrak{M}_\beta$ $(\beta > \gamma + 1/2)$; b)

$$\sum_i d_i^p(H) < \infty \qquad (p < \gamma^{-1}),$$

then the system of eigenvectors and associated vectors of each of the pencils (1) and (2) is n-fold complete in $\mathfrak{B}$. Moreover, for any set of vectors $\{f_\nu\}_0^{n-1}$ satisfying the conditions

$$(I - T_0) f_\nu \in \mathfrak{R}(H) \qquad (\nu = 0, 1, \dots, n-1),$$

the n-fold expansions in the system of eigenvectors and associated vectors of the pencil (2) are summable by the Abel method of order α, where

$$\gamma^{-1} > \alpha \ge (\beta - 1/2)^{-1}$$

under condition a), and

$$\gamma^{-1} > \alpha > p$$

under condition b).

Remark 3. If $T_0 = HB_0$ $(B_0 \in \mathfrak{R})$, then, evidently, the conditions $(I - T_0) f_\nu \in \mathfrak{R}(H)$ are equivalent to the conditions $f_\nu \in \mathfrak{R}(H)$. We also note that, in the case $\alpha = 1$, the last assertion of Theorem 3 is true for any set $f_\nu \in \mathfrak{B}$ $(\nu = 0, 1, \dots, n-1)$, and not only for the pencil (2), but also for the pencil (1).

Remark 4. If, in the hypotheses of Theorem 3, the restriction on the spectrum and the resolvent of the operator H is replaced by the following: the spectrum of the operator H lies in the angles

$$\Lambda_\nu\{z : \ |\arg z - 2\pi\nu/n| \le \gamma\pi/2n\} \qquad (\nu = 0, 1, \dots, n-1),$$

and

$$\|(H - \lambda I)^{-1}\| \le C\,[\min_\nu \rho(\lambda, \Lambda_\nu)]^{-1} \qquad \left(\lambda \notin \bigcup_{\nu=0}^{n-1} \Lambda_\nu\right),$$

then n-fold completeness of the system of eigenvectors and associated vectors of each of the pencils (3) and (4) holds.

2. Henceforth, to the end, a Hilbert space $\mathfrak{H}$ is considered.

We do not state the corollaries of Theorem 3 and Remark 4 for the pencils (1)–(4) with an operator H for which the values of the quadratic form $(H\varphi, \varphi)$ are situated in a certain angle. We note that the completeness theorems obtained in this way admit a certain strengthening based on assumptions on the connection between the spectra of the Hermitian components of the Volterra operator from (9) and (10) (cf. (^{4})).

The following two theorems, like Theorems 1, are proved by reduction to the case $n = 1$, for which they were established by V. I. Macaev (^{11}).

Theorem 4. Let $H \in \mathfrak{S}_\infty$ be a complete self-adjoint operator, $p \geq 0$, and

$$T_\nu = H_\nu^{pB}, \qquad B_\nu \in \mathfrak{S}_\infty \quad (\nu = 0, 1, \dots, n-1).$$

If

$$\lim_{k\to\infty} F_p(s_k(H)) \sum_{j=1}^{k} j^{-1} s_j(B_\nu) = 0 \qquad (\nu = 0, 1, \dots, n-1),$$

where

$$F_p(t) = t^p$$

for $p > 0$ and

$$F_0(t) = -|\ln t|^{-1},$$

then the system of eigenvectors and associated vectors of each of the pencils (1) and (2) is n-fold complete in $\mathfrak{H}$.

Theorem 5. Let $H \in \mathfrak{S}_\infty$ be a complete self-adjoint operator, $p \geq 0$, and

$$T_\nu = H_\nu^{pB}, \qquad B_\nu \in \mathfrak{S}_\infty \quad (\nu = 0, 1, \dots, n-1).$$

If

$$\sum_{k=1}^{\infty} k^{-1} s_k(HT_\nu) < \infty; \qquad \lim_{k\to\infty} F_p(t_k^{(\nu)}) \sum_{j=1}^{k} j^{-1} s_j(B_\nu) = 0 \qquad (\nu = 0, 1, \dots, n{-}1),$$

where

$$t_k^{(\nu)} = \sum_{j=k}^{\infty} j^{-1} s_j(HT_\nu),$$

then the system of eigenvectors and associated vectors of each of the pencils (1) and (2) is n-fold complete in $\mathfrak{H}$.

3. Theorem 6. Let $H \in \mathfrak{S}_\infty$ be a complete self-adjoint operator, $p > 0$, $T_\nu = H_\nu^{pB}$ (or $T_\nu = B_\nu H^p$), $\nu = 0, 1, \dots, n-1$. If at least one of the following two conditions is satisfied:

$$\text{a)} \quad \lim_{r\to\infty} r^{-p} n(r, H) = 0; \qquad B_\nu \in \mathfrak{R} \quad (\nu = 0, 1, \dots, n-1);$$

$$\text{b)}\quad \lim_{r\to\infty} r^{-p} n(r,H) < \infty; \qquad B_\nu \in \mathfrak{S}_\infty \quad (\nu = 0, 1, \dots, n-1),$$

then, for any collection of vectors $f_\nu \in \mathfrak{H}$ $(\nu = 0, 1, \dots, n-1)$, the n-fold expansions with parentheses in the system of eigenvectors and associated vectors of each of the pencils (1) and (2) converge unconditionally.

Theorem 6 is established analogously to Theorem 1 (on the basis of Theorem 3 from (12)). Using a somewhat different method, one can obtain certain propositions on the convergence of multiple expansions for the pencils (3) and (4). We give one of them.

Theorem 7. Let $H \in \mathfrak{S}_\infty$ be a complete normal operator, the operator H^n self-adjoint, $p > 0$, and $r(\nu) = n - \nu - 1 - p$ for $p \le 1$, $r(\nu) = \max(n - \nu, p)$ for $p > 1$ $(\nu = 0, 1, \dots, n-1)$. If $T_\nu = H^{r(\nu)} B_\nu$ $(\nu = 0, 1, \dots, n-1)$ and at least one of conditions a), b) of Theorem 6 is satisfied, then, for any collection of vectors $f_\nu \in \mathfrak{R}(H^{n-\nu-1})$ $(\nu = 0, 1, \dots, n-1)$, the n-fold expansions with parentheses in the system of eigenvectors and associated vectors of the pencil (4) converge unconditionally.

Remark 5. The requirement of completeness of the operator H in Theorem 7 can be omitted if the condition $f_{n-1} \in \mathfrak{R}(H)$ is added.

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