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Abstract

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MATHEMATICS

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ON SERIES CLOSE TO DIRICHLET SERIES

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Let $\{\lambda_n\}$ be a sequence of positive numbers satisfying the condition:

$$H > \lambda_{n+1} - \lambda_n > h > 0. \tag{1}$$

Definition 1. A series close to a Dirichlet series, or, briefly, a D-series, will mean a series

$$f(z) = \sum_{n=1}^{\infty} a_n \varphi_n(z), \tag{2}$$

if the following conditions are satisfied:

a) The functions $\varphi_n(z)$ are regular in some bounded domain Q, have no zeros in Q, and satisfy in Q the relations

$$\varphi_n(z) = \varphi_0(z) \exp[-\lambda_n z + u(n, z)], \tag{3}$$

$u(n, z) = o(n)$ uniformly with respect to z;

$$\varphi_{n+m}(z) = \varphi_n(z) \exp[-(\lambda_{n+m} - \lambda_n)z + v(n, m, z)], \tag{4}$$

$v(n, m, z) \to 0$ (as $n \to \infty$ and fixed m) uniformly with respect to z, and $v(n, m, z) = o(m)$ uniformly with respect to n and z.

b) Q contains some part L of the boundary of the domain of convergence of series (2).

In view of (1)–(3), L is situated on a vertical straight line.

Examples of D-series:

$$\sum_{n=1}^{\infty} a_n (z - z_0)^{\alpha_n} \exp(-\lambda_n z),$$

where $\{\lambda_n\}$ satisfies (1), $\alpha_{n+1} - \alpha_n \to 0$ $(n \to \infty)$,

$$0 < \overline{\lim} \sqrt[n]{|a_n|} < \infty;$$

a lacunary series in the Chebyshev-Hermite polynomials $H_{n_k}(iz)$, where

$$\sqrt{n_{k+1}} - \sqrt{n_k} > h > 0;$$

some series in other eigenfunctions of differential equations; the series

$$\sum_{k=1}^{\infty} a_k z^{\alpha_k} \exp(-\lambda_k z) H_{n_k}(iz)$$

and so on.

Definition 2. A D-series for which L lies on the imaginary axis will be called **reduced**.

A general D-series is reduced to a reduced one by a linear change of the variable z. From (1)–(3) it follows that for a reduced series

$$\overline{\lim} \sqrt[n]{|a_n|} = 1.$$

In this case (see (1′) or (2′)) there exists a sequence of positive numbers $\{q_n\}$ possessing the following properties:

$$q_{n+1} q_n^{-1} \to 1 \quad (n \to \infty); \tag{5}$$

$$|a_n| < C q_n, \tag{6}$$

$$|a_{n_k}| = q_{n_k}$$

for some infinite sequence of indices $\{n_k\}$.

(7)

Definition 3. Any sequence $\{n_k\}$ for which there exists $\{q_n\}$ such that (5), (6), (7) are satisfied will be called a sequence of principal indices of the reduced series.

Some theorems known for Dirichlet series we shall prove here for D-series.

Theorem 1. *Let the sum of the reduced series be regular at the point z_0, $z_0 \in L$. If $q_n \varphi_n(z_0) \to 0$ as $n \to \infty$, then the series converges at the point z_0.*

Theorem 2 (on gaps). *Let there exist such a sequence of principal indices $\{n_k\}$ of the reduced series and such a sequence of integers $\{m_k\}$ ($m_k \to \infty$ as $k \to \infty$), that $a_{n_k+\nu} = 0$, $\nu = 1, 2, \dots, m_k$. Then every point of L is singular for the sum of the series.*

The theorems are proved with the aid of the following lemmas.

Lemma 1. *Let the sum of the reduced series be regular in some closed domain T, with $T \subset Q$ and T containing within itself a segment of the imaginary axis. Then the family of functions $\{f_n(z)\}$, where*

$$f_n(z) = \left[f(z) - \sum_{j=1}^{n} a_j \varphi_j(z) \right] q_n^{-1} \varphi_n^{-1}(z), \tag{8}$$

is bounded in T.

Proof. Suppose, to the contrary, that the family $\{f_n(z)\}$ is not bounded in T. Then, if we denote $p_n = \max_T |f_n(z)|$, there is a sequence of indices $\{n_l\}$ such that

$$p_n \to \infty, \quad \text{when } n \text{ runs through } \{n_l\}. \tag{9}$$

Let $\tilde{f}_n(z) = f_n(z) p_n^{-1}$; then $\max_T |\tilde{f}_n(z)| = 1$. From $\{n_l\}$ one can extract a subsequence $\{n'_l\}$ such that $\tilde{f}_n(z) \to \tilde{g}(z)$ uniformly in T, when n runs through $\{n'_l\}$; $\tilde{g}(z)$ is regular in T and

$$\max_T |g(z)| = 1. \tag{10}$$

Let $T_1 = \{T \cap \operatorname{Re} z > \delta\}$, where $\delta > 0$ and is sufficiently small. For $z \in T_1$ we replace the square bracket in (8) by the remainder of the series (2). Owing to (1), (4), (5), (6), (9), from $\{n_l\}$ one can, by means of a diagonal process, extract a subsequence $\{n''_l\}$ along which termwise passage to the limit is possible in the resulting series for $\tilde{f}_n(z)$. A Dirichlet series for $\tilde{g}(z)$ is obtained. But, in view of (9), all its coefficients are equal to zero, and $\tilde{g}(z) \equiv 0$, which contradicts (10). Lemma 1 is proved.

Theorem 1 is a consequence of Lemma 1.

Lemma 2. *Under the conditions of Lemma 1, every limit function $g(z)$ of the family $\{f_n(z)\}$ is representable in the half-plane* $\operatorname{Re} z > 0$ *by the Dirichlet series*

$$g(z) = \sum_{m=1}^{\infty} b_m \exp(-\mu_m z), \tag{11}$$

and in the half-plane $\operatorname{Re} z < 0$ *by the Dirichlet series*

$$g(z) = b_0 + \sum_{m=1}^{\infty} b_{-m} \exp(\mu_{-m} z); \tag{12}$$

$$H \geq \mu_{\pm(m+1)} - \mu_{\pm m} \geq h > 0, \quad \mu_{\pm 1} \geq h > 0; \tag{13}$$

$$|b_{\pm m}| \leq C, \tag{14}$$

where H and h are constants from (1), and C is from (6).

Proof. As in the proof of Lemma 1, one can extract such a subsequence of indices along which termwise passage to the limit in T_1 in (8) is possible, which gives (11), and from it—such a subsequence along which termwise passage to the limit in $T_2 = \{T \cap \operatorname{Re} z < -\delta\}$ is possible, which gives (12).

Lemma 3. *If, under the conditions of the preceding lemmas, in (8) n runs in advance through only some subsequence of principal indices $\{n_k\}$, then $g(z)$ has at least one singular point on each segment of the imaginary axis of length greater than $2\pi h^{-1}$, and in (12) $|b_0| = 1$.*

Proof. In (12)

$$b_0 = -\lim(a_n q_n^{-1}) \quad (n \in \{n_k\});$$

in view of (7), $|b_0| = 1$. If, on the contrary, $g(z)$ were regular on a segment of the imaginary axis of length greater than $2\pi h^{-1}$, then, by Pólya' s theorem for Dirichlet series, the boundary of convergence of the series (11) would lie to the left of the imaginary axis. It would follow from (11) and (12) that $g(z)$ is bounded in the whole plane, i.e. $g(z) \equiv \text{const}$. But from (11), as $\operatorname{Re} z \to \infty$, one obtains $g(z) = 0$, while from (12), as $\operatorname{Re} z \to -\infty$, $g(z) = b_0$. The contradiction obtained proves Lemma 3.

Proof of Theorem 2. Suppose, to the contrary, that $f(z)$ can be analytically continued to the left through some segment belonging to L. Then $f(z)$ is regular in some domain T of the kind described in Lemma 1. Applying Lemmas 1-3, we obtain in (12) $|b_0| = 1$. But, owing to the omissions in the series (2), all b_m in (11) vanish, and $g(z) \equiv 0$, which is incompatible with $|b_0| = 1$. The contradiction obtained proves the theorem.

Theorem 3. *Every segment belonging to L and having length greater than $2\pi h^{-1}$ contains at least one singular point of the sum of the D-series.*

The proof of Theorem 3 is analogous to the preceding one.

Some other theorems on Dirichlet series can also be generalized to D-series.

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Note: Figure translations are in progress. See original paper for figures.