OPTIMAL MAGNETO-HYDRODYNAMIC GENERATOR

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Abstract

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OPTIMAL MAGNETOHYDRODYNAMIC GENERATOR

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Optimal generation of power with respect to the load parameter K in a magnetohydrodynamic generator (MHDG) with a constant channel cross-sectional area and with constant magnetic and electric fields was investigated in ([1]). Some other special cases of optimization of MHDG operating regimes (mainly non-functional optimization) were considered in ([2–5]). In the present work a general solution of this problem is given. It is noteworthy that it is obtained in finite form and is quite simple.

The formulation of the problem coincides with ([6]) and is determined by the variables: gas velocity $\mathbf{v}$ $(v, 0, 0)$, electric field intensity $\mathbf{E}$ $(0, -E, 0)$, magnetic induction of the external field $\mathbf{B}$ $(0, 0, -B)$, current density $\mathbf{j}$ $(0, j, 0)$, gas density ρ, pressure p, temperature T, enthalpy w, entropy s. The motion of the conducting gas with conductivity σ takes place along the x-axis of a channel of height y and width z, so that the channel cross-sectional area is $S = yz$. The equations in the one-dimensional approximation, neglecting the induced magnetic field, viscosity, and thermal conductivity, have the form

$$\rho v S = m, \qquad \rho v \frac{dv}{dx} + \frac{dp}{dx} = -jB, \qquad \rho v T \frac{ds}{dx} = \frac{j^2}{\sigma}, \tag{1}$$

$$p = \rho \frac{R}{\mu} T, \qquad j = \sigma(vB - E),$$

where $m = \text{const}$ is the mass flow rate, R is the gas constant, and μ is the molecular weight. Using the thermodynamic relation $T\,ds = dw - \frac{1}{\rho}\,dp$, the energy equation can be written in the form

$$\rho v \frac{d}{dx}\left(w + \frac{v^2}{2}\right) = -jE.$$

Introducing the power

$$N = \int_0^x jES\,dx,$$

we obtain the efficiency of the MHDG

$$\eta = \frac{N}{m(w + v^2/2)_1}$$

(the subscript 1 refers to the initial section of the channel). As the independent variable we introduce η $(0 < \eta < 1)$, which makes it possible to integrate the energy equation at once. We pass to dimensionless variables $\varphi = \varphi/\varphi_1$, except for $\bar{x} = x/(\rho v/\sigma B^2)_1$ (in what follows, conversely, bars are placed over dimensional quantities). After the necessary substitutions, the equation of motion for an ideal gas has the form

$$f \equiv \gamma \mathrm{M}_1^2 v \frac{dv}{d\eta} + S \frac{d}{d\eta}\left(\frac{T}{Sv}\right) + \frac{\gamma}{\gamma - 1}\left(1 + \frac{\gamma - 1}{2}\mathrm{M}_1^2\right)\frac{1}{1 - K_1}\frac{B}{E} = 0, \qquad (2)$$

where

$$T = \left(1 + \frac{\gamma - 1}{2}\mathrm{M}_1^2\right)(1 - \eta) - \frac{\gamma - 1}{2}\mathrm{M}_1^2 v^2 \qquad (3)$$

is the energy equation; $K_1 = 1 - (\bar{E}/\bar{v}\bar{B})_1$ $(0 < K_1 < 1)$ is the initial load parameter, completing the electrical efficiency to 1; M_1 is the Mach number, $\gamma = c_p/c_v$.

The problem has been reduced to integration of one equation (2) with 4 functions v, S, B, and E. To determine them it is necessary to formulate a variational problem.

A general proposition is put forward: optimization of an arbitrary irreversible process is achieved by minimizing the integral rate of entropy production under admissible functional constraints. The principle of the minimum rate of entropy production was formulated by I. Prigogine (see, for example, (7)) and proved by him for stationary processes in so-called linear thermodynamics of irreversible processes on the basis of the Onsager relations. The well-known principle of minimum energy dissipation (in particular, Joule dissipation) is a particular (isothermal) formulation of the principle under consideration. Let us note that for one-dimensional flows, in view of the continuity equation $S\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v S) = 0$, one has

$$\int \rho \frac{ds}{dt}\, dV = \rho v S \Delta s + \int \frac{\partial}{\partial t}(\rho s)\, dV \qquad (\Delta s = s - s_1),$$

which for a stationary process leads to

$$\int \rho \frac{ds}{dt}\, dV = m \Delta s,$$

so that in this particular case the principle under consideration reduces to minimization of the final value of the entropy.

In the present work the principle of the minimum integral rate of entropy production is used to carry out optimization of an MHD generator. As the generalized functional, a linear combination of the final value of the entropy and the volume of the MHD generator is taken. From the standpoint of the reciprocity principle this is equivalent also to minimization of the volume of the MHD generator at a prescribed power.

In view of the presence of the nonholonomic constraint (2), we obtain a problem for the conditional extremum of the functional

$$J = \Delta s + \nu \int_0^\eta (F + \lambda f)\, d\eta, \qquad \lambda = \lambda(\eta) \text{ and } \nu = \text{const}$$

—undetermined Lagrange multipliers. For the reduced volume of the MHD generator we have the relation

$$V(1-K_1)\,\frac{(\gamma-1)\mathrm{M}_1^2}{1+\dfrac{\gamma-1}{2}\mathrm{M}_1^2} = \int_0^\eta \frac{d\eta}{\delta e E} = \int_0^\eta F\, d\eta, \tag{4}$$

where $V = \bar{V}(\bar{\sigma}\bar{B}^2)/m$ is the dimensionless volume of the MHD generator (the generalized parameter of magnetohydrodynamic interaction (1)), $e = vB - (1 - K_1)E$. The Euler equations for the functions $\varphi(S, B, E, v)$ have the form

$$F_\varphi/\lambda + f_\varphi - (f_{\varphi'})' = f_\varphi \lambda'/\lambda \tag{5}$$

(letter subscripts denote differentiation with respect to φ and φ'; the prime denotes differentiation with respect to η).

We shall further assume the conductivity to be constant ($\sigma = 1$). From (5), with $\varphi \equiv S$, we obtain $\lambda = \lambda_1 v/T$, where λ_1 is a constant of integration. From (5), with $\varphi \equiv B$, we obtain

$$(vBK)^2/T = K_1^2 \left(\lambda_1 = \frac{\gamma-1}{\gamma}\,\frac{1-K_1}{K_1^2}\left(1+\frac{\gamma-1}{2}\mathrm{M}_1^2\right)^{-1}\right). \tag{6}$$

Putting further $\varphi \equiv v$, we obtain from (5) the equation

$$(\mathrm{M}^2-1)\left(\ln\frac{v}{T}\right)' - [1+(\gamma-1)\mathrm{M}^2](\ln S)' + \frac{\gamma}{\gamma-1}\,\frac{1+\dfrac{\gamma-1}{2}\mathrm{M}_1^2}{(1-K)T} = 0, \tag{7}$$

which, together with the transformed equation (2),

$$\gamma M^2(\ln v)' - \left(\ln \frac{v}{T}\right)' - (\ln S)' + \frac{\gamma}{\gamma-1}\,\frac{1+\frac{\gamma-1}{2}M_1^2}{(1-K)T} = 0 \tag{8}$$

and the continuity equation $\rho v S = 1$, gives the adiabat $T = \rho^{\gamma-1}$, i.e., in this case the stationarity condition for the rate of entropy generation (6) is satisfied trivially: this rate is equal to 0 (and with it, so is the current). Thus, simultaneous variation of the functional with respect to S and B is impossible. Leaving in this case the magnetic field constant ($B = 1$), set $\varphi \equiv E$ and obtain from (5)

$$\frac{(vK)^2}{(1-2K)T} = \frac{K_1^2}{1-2K_1} \quad \left(\lambda_1 = \frac{\gamma-1}{\gamma}\,\frac{(1-K_1)(1-2K_1)}{K_1^2}\left(1+\frac{\gamma-1}{2}M_1^2\right)^{-1}\right). \tag{9}$$

Next, setting $\varphi \equiv v$, we obtain from (5) an equation analogous to (7), which together with (8) gives

$$\left(\ln \frac{T}{\rho^{\gamma-1}}\right)' = \frac{\gamma}{\gamma-1}\,\frac{1+\frac{\gamma-1}{2}M_1^2}{(1-K)T}\,\frac{1}{M^2}\left(\frac{1}{1-2K}-1\right).$$

Comparison of this equation with the reduced form of the energy equation (1)

$$\left(\ln \frac{T}{\rho^{\gamma-1}}\right)' = \frac{\gamma}{\gamma-1}\,\frac{1+\frac{\gamma-1}{2}M_1^2}{(1-K)T}\,(\gamma-1)K$$

gives the relation

$$\frac{\gamma-1}{2}M^2(1-2K) = 1. \tag{10}$$

Relations (9) and (10) are equivalent to the following basic results: $K = K_1$, $M = M_1$, i.e., in an optimal MHD generator the conversion into electrical energy occurs uniformly along the channel, while the conversion itself proceeds uniformly with respect to both the thermal and mechanical forms of energy. Taking into account the required constancy of the magnetic field, one may say that, under the general optimization carried out, the MHD generator is a generator of the VKM type in the presence of the relation (10) between K and M (the meaning of this relation is clarified below). The presence of the relation (10) means that the optimal MHD generator can only be supersonic

$$\left(M_{\min} = \sqrt{\frac{2}{\gamma - 1}} > 1\right).$$

The constancy of K and M immediately makes it possible to obtain the last integral, and with it the remaining relations ($n = \gamma M^2 - 2$):

$$T = 1 - \eta, \qquad v = E = \sqrt{T}, \qquad \rho = v^n, \qquad S = (\rho v)^{-1}, \qquad p = \rho T. \quad (11)$$

From (4) and (11) one obtains an explicit expression for S as a function of x

$$S = \left(1 - \frac{x}{X}\right)^{-1} \quad \left(X = \frac{2}{K(1-K)} \frac{1 + \frac{\gamma-1}{2} M^2}{(\gamma - 1) M^2} \frac{1}{n+1}\right), \qquad (12)$$

so that the profiling of the cross-sectional area of the channel of the optimal MHD generator is performed along a hyperbola and, for a given M (or K), the optimal MHD generator has a length not exceeding X. Using (12), we determine the volume of the MHD generator

$$V = \int_0^x S\,dx = X \ln S. \qquad (13)$$

Finally, the isentropic efficiency coefficient ξ in the form (6) ($\eta < \xi < 1$)

$$\xi = \eta \left[1 - e^{-\frac{\Delta s}{c_p}} (1 - \eta)\right]^{-1} = \eta \left[1 - T^{\frac{\gamma-1}{\gamma}\left(\frac{n}{2}+1\right)}\right]^{-1}. \qquad (14)$$

We note that the process in the channel takes place along the polytrope $p = \rho^\Gamma$, where $\Gamma = (n+2)/n$, and, in view of (10), $\gamma > \Gamma > 1$ ($2/(\gamma - 1) < n < \infty$).

To clarify the meaning of relation (10), let us solve the ordinary problem for $B, K, M = \text{const}$. The solution, naturally, is obtained in the same way, but with a different value

$$n = \frac{1}{1-K}\left[\frac{2}{\gamma - 1} + \left(2 + \gamma M^2\right) K\right].$$

If one now poses the ordinary problem for the conditional extremum (minimum) of the expression $I = \nu_1 \Delta s + \nu_2 V$ as a function of M and K, it turns out that the minimum of I is realized precisely when relation (10) is satisfied. Thus relation (10) ensures the minimum of the final value of the entropy for a given volume of the MHDG or, according to the reciprocity principle, the minimum volume of the MHDG for a given final value of the entropy.

The transversality condition at the free end of the extremal has the form

$$\delta J = \delta\Delta s + \nu\lambda\left(f_v \delta v + f_S \delta S\right) = 0 \tag{15}$$

and is satisfied for a proper value of ν in any section of the channel. This means that any segment of the channel of an optimal MHDG is optimal (and the beginning and end of the segment are arbitrary). It is interesting that the channel shape (12) can, in this connection, be obtained from a general functional requirement: the ratio of areas S_2/S_1 is a function only of the parameter of magnetohydrodynamic interaction $(\bar{\sigma}\bar{B}^2\bar{S})|\bar{x}|/\dot{m} \sim S_1 x$.

Indeed, the following functional equations are valid:

$$S_2/S_1 = f\left(S_1(x_2 - x_1)\right), \quad S_3/S_1 = f\left(S_1(x_3 - x_1)\right), \quad S_3/S_2 = f\left(S_2(x_3 - x_2)\right),$$

where f is an unknown function (obviously, $f(0) = 1$). This system reduces to the functional equation $f(x)f[yf(x)] = f(x+y)$, where x and y are independent. The only solution of the latter equation is the function inverse-linear, $f(x) = (1 - f'(0)x)^{-1}$, i.e., expression (12).

In a concrete realization of an optimal MHDG, special cases are possible: 1) voltage $U = 1$, 2) $z = 1$, 3) $y = 1$. For $U = 1$ we have $y = U/E = v^{-1} < z = \rho^{-1} = v^{-n}$; for $z = 1$, $y = S$ and $U = Ey = v^{-m}$—the voltage increases along the channel; for $y = 1$, $z = S$ and $U = Ey = v$—the voltage decreases along the channel. For $U = 1$, obviously,

$$K = (1 + \bar{R}/\bar{r})^{-1},$$

where $\bar{R}$ and $\bar{r}$ are, respectively, the external and internal resistance of the MHDG; in this case

$$\bar{r}^{-1} = \left(m/(\bar{y}\bar{B})^2\right)_1 \frac{n+1}{2} X\eta.$$

To minimize the volume, as is seen from (4), the gas conductivity must, naturally, be as large as possible. The dimensional length of the MHDG, determined by the scale $(\rho\bar{v}/\bar{\sigma}\bar{B}^2)_1$, will be the smaller, the smaller the specific mass flow rate and the greater the gas conductivity and the magnetic induction of the external field.

For the quasi-one-dimensional approximation adopted here to be valid, it is necessary that

$$\theta_1 = \frac{d\sqrt{\bar{s}}}{dx} = \left(\frac{\sqrt{\bar{s}}}{\rho v/\bar{\sigma} B^2}\right)_1 \frac{s^{3/2}}{2X}$$

be small, or that

$$\theta_2 = \frac{d\bar{y}}{dx} = \left(\frac{\bar{y}}{\rho v/\bar{\sigma}B^2}\right)_1 \frac{s^2}{X}$$

be small.

In considering possible particular variational problems, it should be borne in mind that, apart from the impossibility of simultaneous variation of S and B, as is not difficult to show, it is impossible to vary E and B simultaneously. Thus there exist only the following particular problems: 1) $B = E = 1$, 2) $B = v = 1$, 3) $B = S = 1$, 4) $E = S = 1$.

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Note: Figure translations are in progress. See original paper for figures.