THE COMMUTATION FUNCTION OF A SCALAR FIELD IN A TWO-DIMENSIONAL MODEL OF PSEUDO-RIEMANNIAN SPACE-TIME

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Abstract

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THE COMMUTATION FUNCTION OF A SCALAR FIELD IN A TWO-DIMENSIONAL MODEL OF PSEUDO-RIEMANNIAN SPACE-TIME

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In recent years works have appeared ($^{1-3}$) devoted to the construction of quantum field theory in an external gravitational field. However, in these works the gravitational field is assumed either to be weak or to satisfy quite strong special requirements. In the present work a two-dimensional model of an arbitrary pseudo-Riemannian space-time is considered, and for this case an explicit expression for the commutation function of a scalar field is found.

Let us first consider the true case of four-dimensional space-time. In accordance with the flat case, it is required to solve the Cauchy problem for the Klein-Gordon equation

$$\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\alpha}\left(\sqrt{-g}\,g^{\alpha\beta}\frac{\partial\psi}{\partial x^\beta}\right)+m^2\psi=0 \tag{1}$$

under the condition that on some space-like hypersurface Σ the function $\psi(x)$ and its derivative in the direction normal to Σ are prescribed. On the hypersurface Σ the function $\psi(x)$ is an operator subject to the conditions

$$[\psi(M_1),\,\psi(M_2)]_{M_1,M_2\in\Sigma}=0,$$

$$[n^\alpha(M_1)\psi_\alpha(M_1),\,n^\beta(M_2)\psi_\beta(M_2)]_{M_1,M_2\in\Sigma}=0, \tag{2}$$

$$\int f(M)[\psi(M_1),\,\psi_\alpha(M)]_{M\in\Sigma}\,d\sigma^\alpha(M)=if(M_1),$$

where $\psi_\alpha=\partial\psi/\partial x^\alpha$, $n^\alpha(M)$ is the normal to Σ, and $f(M)$ is an arbitrary function*.

Having solved this Cauchy problem, one can calculate the commutation function

$$D(M_1, M_2) = i[\psi(M_1),\ \psi(M_2)]; \tag{3}$$

here M_1 and M_2 are arbitrary points of space-time.

In the two-dimensional case the stated problem can be solved explicitly by Riemann' s method. In this case the metric form of space-time can always be reduced to the form

$$ds^2 = 4a^2(x,y)\,dx\,dy. \tag{4}$$

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* In the general case of an $(n+1)$-dimensional space,

$$\psi_\alpha d\sigma^\alpha = \sqrt{-g}\begin{vmatrix} \psi^0 & \psi^1 & \dots & \psi^n \\ d_1x^0 & d_1x^1 & \dots & d_1x^n \\ \cdot & \cdot & \cdot & \cdot \\ d_{nx}^0 & d_{nx}^1 & \dots & d_{nx}^n \end{vmatrix}, \qquad \psi^\alpha = g^{\alpha\beta}\psi_\beta.$$

The coordinates x, y are called isotropic, and in them the "future" with respect to the point $M_0(x_0, y_0)$ is determined by the conditions

$$(x-x_0)(y-y_0) > 0, \qquad x+y > x_0+y_0. \tag{5}$$

The Klein–Gordon equation in isotropic coordinates is written in the following form:

$$\frac{\partial^2\psi}{\partial x\,\partial y} + a^2(x,y)m^2\psi = 0. \tag{6}$$

We are required to find a solution of equation (6) under the condition that on the curve

$$y = \mu(x), \qquad \mu' < 0 \tag{7}$$

the quantities $\psi = \varphi(x)$ and

$$[\partial\psi/\partial x - \mu'\partial\psi/\partial y]_{y=\mu(x)} = \pi(x)$$

are prescribed. The operators $\varphi(x)$ and $\pi(x)$, in accordance with (2), obey the commutation relations

$$[\varphi(x_1), \varphi(x_2)] = 0, \qquad [\pi(x_1), \pi(x_2)] = 0,$$
$$[\varphi(x_1), \pi(x_2)] = i\delta(x_1 - x_2), \tag{8}$$

Fig. 1

Figure 1: Fig. 1

since on the curve (7) $\psi_\alpha d\sigma^\alpha = \pi(x)\, dx$.

Fig. 1

Riemann' s method is based on Green' s formula

$$A = \oint_\Sigma d\sigma^\alpha \left(v\psi_\alpha - \psi v_\alpha\right) =$$
$$= \int_V dV \left(v\nabla_\alpha \psi^\alpha - \psi \nabla_\alpha v^\alpha\right); \tag{9}$$

here ∇_α is the symbol of covariant differentiation and dV is the covariant volume element of the domain V. If ψ and v obey the Klein-Gordon equation, then the integrand in the last integral becomes zero, and consequently, for such ψ and v,

$$A = \oint_\Sigma d\sigma^\alpha \left(v\psi_\alpha - \psi v_\alpha\right) = 0. \tag{10}$$

Let us apply this formula in the two-dimensional case to the contour $P_0Q_0M_0$, shown in Fig. 1, where $M_0(x_0, y_0)$ is an arbitrary point. We have

$$A = \int_{Q_0}^{M_0} dy \left(v\psi_y - \psi v_y\right)\big|_{x=x_0} + \int_{M_0}^{P_0} dx \left(\psi v_x - v\psi_x\right)\big|_{y=y_0} +$$
$$+ \int_{P_0}^{Q_0} dx \left(\psi v_n - v\psi_n\right)\big|_{y=\mu(x)} = 0, \tag{11}$$

where $a_n = dx - \mu' dy$ is denoted.

Choose as $v = v(x, y; x_0, y_0) = v_0(x, y)$ the Riemann function, which is a solution of Goursat' s problem:

$$\frac{\partial^2 v_0(x,y)}{\partial x\, \partial y} + a^2(xy) m^2 v_0(xy) = 0, \qquad v_0\big|_{x=x_0} = v_0\big|_{y=y_0} = 1. \tag{12}$$

Then formula (11) gives

$$\psi(M_0) = \frac{1}{2}\psi(Q_0) + \frac{1}{2}\psi(P_0) + \frac{1}{2}\int_{P_0}^{Q_0} dx \left(v_0\psi_n - \psi v_{0n}\right)\big|_{y=\mu(x)}. \tag{13}$$

Thus, we obtain an expression for a function satisfying the Klein–Gordon equation in terms of its value and the value of its normal derivative on the curve $y = \mu(x)$. In particular, the field operator at the point $M_0(x_0, y_0)$ is equal to

$$\psi(M_0) = \frac{1}{2}\varphi(x_0) + \frac{1}{2}\varphi(\mu^*(y_0)) +$$

$$+\frac{1}{2}\int_{\mu^*(y_0)}^{x_0} dx\{v_0(x,\mu(x))\pi(x) - v_{0n}(x,\mu(x))\varphi(x)\}, \tag{14}$$

where $x = \mu^*(y)$ is the function inverse to the function $y = \mu(x)$.

Putting in (13) $\psi(M_0) = v(M_0, M_1)$, we obtain an identity for the function $v(M_0, M_1)$:

$$2v(M_0; M_1) = v(Q_0; M_1) + v(P_0; M_1) +$$

$$+\int_{P_0}^{Q_0} dx \left|\begin{matrix} v(M;M_0) & v(M;M_1) \\ v_n(M;M_0) & v_n(M;M_1)\end{matrix}\right|_{y=\mu(x)}. \tag{15}$$

With the aid of formula (11) it is not difficult to prove one more identity for the function $v(M_1; M_2)$. Let in (11) the point M_0 have coordinates $x_0 = x_2$, $y_0 = y_1$. Then the point P_0 coincides with the point P_1, and the point Q_0 with the point Q_2. Putting further $\psi(M) = v(M; M_1)$, $v(M) = v(M; M_2)$, we find

$$v(Q_2; M_1) - v(P_1; M_2) = \int_{P_1}^{Q_2} dx \left|\begin{matrix} v(M;M_1) & v(M;M_2) \\ v_n(M;M_1) & v_n(M;M_2)\end{matrix}\right|_{y=\mu(x)}. \tag{16}$$

From (15) and (16) it follows that the Riemann function is symmetric:

$$v(M_1; M_2) = v(M_2; M_1). \tag{17}$$

The results obtained make it possible to find the commutation function $D(M_1, M_2)$. In computing it one has to commute operators of the form:

$$A_i = \varphi(a_i) = \varphi(b_i) + \int_{a_i}^{b_i} dx\{p_i(x)\pi(x) + q_i(x)\varphi(x)\} \quad (i = 1, 2); \tag{18}$$

without loss of generality one may assume that $b_i > a_i$. Three cases are possible.

In the first case the intervals $[a_1, b_1]$ and $[a_2, b_2]$ do not overlap, and then, obviously, $[A_1, A_2] = 0$.

In the second case these intervals overlap partially; taking $a_2 > a_1$, we obtain

$$i[A_1(M_1), A_2(M_2)] = p_1(a_2) - p_2(b_1) + \int_{a_2}^{b_1} dx \begin{vmatrix} p_1(x) & p_2(x) \\ q_1(x) & q_2(x) \end{vmatrix}. \tag{19}$$

In the third case one of the intervals, for example, wholly contains the other. Then

$$i[A_1(M_1), A_2(M_2)] = p_1(a_2) + p_1(b_2) + \int_{a_2}^{b_2} dx \begin{vmatrix} p_1(x) & p_2(x) \\ q_1(x) & q_2(x) \end{vmatrix}. \tag{20}$$

Using expressions of the form (19) and (20), obtained by direct commutation of $\psi(M_1)$ and $\psi(M_2)$, and, on the other hand, the identities (15), (16), (17) for the Riemann function, it is not difficult to obtain that

$$D(M_1, M_2) = \frac{1}{2}\varepsilon(M_1, M_2) v(M_1; M_2), \tag{21}$$

where

$$\varepsilon(M_1, M_2) = \begin{cases} 1, & \text{if the point } M_1 \text{ is located in the "future" relative to } M_2; \\ -1, & \text{if the point } M_2 \text{ is located in the "future" relative to } M_1; \\ 0, & \text{if the points } M_1 \text{ and } M_2 \text{ are situated spacelike.} \end{cases}$$

In isotropic coordinates $\varepsilon(M_1, M_2)$ can be expressed through the function

$$\theta(x) = \begin{cases} 1, & x > 0, \\ 0, & x < 0, \end{cases}$$

as follows:

$$\varepsilon(M_1, M_2) = \theta(x_1 - x_2)\theta(y_1 - y_2) - \theta(x_2 - x_1)\theta(y_2 - y_1). \tag{22}$$

Differentiating (21), one can show that not only on the curve $y = \mu(x)$, but also on any spacelike curve $y = \tilde{\mu}(x)$, the operators $\tilde{\pi}(x) = [\partial\psi/\partial x - \tilde{\mu}'(x)\partial\psi/\partial y]_{y=\tilde{\mu}(x)}$ and $\widetilde{\varphi}(x) = \psi(x, \tilde{\mu}(x))$ satisfy the commutation relations (8).

In conclusion, we note that the Riemann function, as follows from (9), is a solution of the integral equation

$$v(x, y; x_0, y_0) = 1 - m^2 \int_{x_0}^{x} d\xi \int_{y_0}^{y} d\eta\, a^2(\xi, \eta) v(\xi, \eta; x_0, y_0). \tag{23}$$

Solving this equation by the method of successive approximations leads to the expression for $v(x, y; x_0, y_0)$ in the form of a convergent series:

$$v(x, y; x_0, y_0) = 1 + \sum_{n=1}^{\infty} (-1)^n m^{2n} \int_{x_0}^{x} d\xi_1 \int_{y_0}^{y} d\eta_1\, a^2(\xi_1, \eta_1) \int_{x_0}^{\xi_1} d\xi_2 \int_{y_0}^{\eta_1} d\eta_2\, a^2(\xi_2, \eta_2) \cdots \cdots \int_{x_0}^{\xi_{n-1}} d\xi_n \int_{y_0}^{\eta_{n-1}} d\eta_n\, a^2(\xi_n, \eta_n). \tag{24}$$

In the flat case $a^2(x, y) = 1$, from (24) we obtain

$$v(x, y; x_0, y_0) = J_0\left(2m\sqrt{(x - x_0)(y - y_0)}\right), \tag{25}$$

where J_0 is the Bessel function. In this case the expression $4(x - x_0)(y - y_0)$ is the square of the distance between the points M and M_0.

We note that for $m^2 = 0$ the results obtained coincide completely with the analogous results for the flat two-dimensional case. This is due to the fact that for $m^2 = 0$ the function $a^2(x, y)$ drops out of equation (9). In other words, in the two-dimensional model the gravitational field does not affect particles with zero rest mass.

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Note: Figure translations are in progress. See original paper for figures.